1.(a) Since there is a positively invariant closed set containing the equilibrium at ($u_e = a+b$, $v_e = \frac{b}{(a+b)^2}$). Thus, in order to show that a closed orbit exists, we show that this set excluding the equilibrium is positively invariant, by choosing a, b so that the equilibrium is an unstable node or focus. Thus, by Poincaré-Bendixson, we have the result.

Jacobian $J = \left.\begin{bmatrix} -1+2uv & u^2 \\ -2uv & -u^2 \end{bmatrix}\right|_{(u_e, v_e)}$

$\therefore \; |\lambda I - J|_e = 0$

$\Rightarrow \lambda^2 - (-u_e^2 - 1 + 2u_e v_e)\lambda + u_e^2 = 0$

For (u_e, v_e) to be an unstable node or focus, we need:

$$-u_e^2 - 1 + 2u_e v_e > 0$$

$$\Rightarrow -(a+b)^2 - 1 + 2(a+b)\frac{b}{(a+b)^2} > 0$$

$$\Rightarrow -(a+b)^2 - 1 + \frac{2b}{(a+b)} > 0 \longleftarrow$$

1 (b)

(i) No, because the positively invariant region must include the boundary of the region bounded by these trajectories, and thus this positively invariant set contains 4 equilibria. Thus Poincaré - Bendixson cannot be used.

(ii) Consider $V(x,y) = -\frac{1}{2}(x^2+y^2) + \frac{1}{2}(xy^2 - \frac{x^3}{3})$

$$\frac{\partial V}{\partial x} = -x + \frac{y^2}{2} - \frac{x^2}{2}$$

$$\frac{\partial V}{\partial y} = -y + xy$$

$\therefore \quad \dot{V}(x,y) = 0$

$\Rightarrow V(x,y) = \text{const.}$ are traces of trajectories of the system.

Consider $V(x,y)$ near the origin.

$V(x,y) = \text{const.}$

$$\Rightarrow -\frac{1}{2}(x^2+y^2) + \frac{1}{2}(xy^2 - \frac{x^3}{3}) = \text{const}$$

Level sets of $V(x,y)$ for small x, y, ie for $x = \varepsilon_1$, $y = \varepsilon_2$

$\Rightarrow \varepsilon_1^2 + \varepsilon_2^2 \simeq \text{const}$

$\Rightarrow$ closed orbits around origin

and, since the region is invariant, these closed orbits are contained in this region

2. $\dot{x} = d_1 a x - d_2 x^2 - d_3 b x$

$\Rightarrow \dot{x}_1 = (d_1 a - d_3 b) x - d_2 x^2$

equilibria:

$\dot{x} = 0 \Rightarrow (d_1 a - d_3 b) x_e = d_2 x_e^2$

$\Rightarrow x_e = 0$ or $x_e = \dfrac{d_1 a - d_3 b}{d_2} =: \dfrac{\mu}{d_2}$

$D_x f_\mu \big|_{x_e} = \mu - 2 d_2 x_e$

$\Rightarrow$ Bifurcation possible at $x_e = 0, \mu = 0$.

$\underline{x_e = 0}$: $D_x f_\mu \big|_{x_e} = \mu$

$\Rightarrow \mu > 0$: Unstable equilibrium at $x_e = 0$

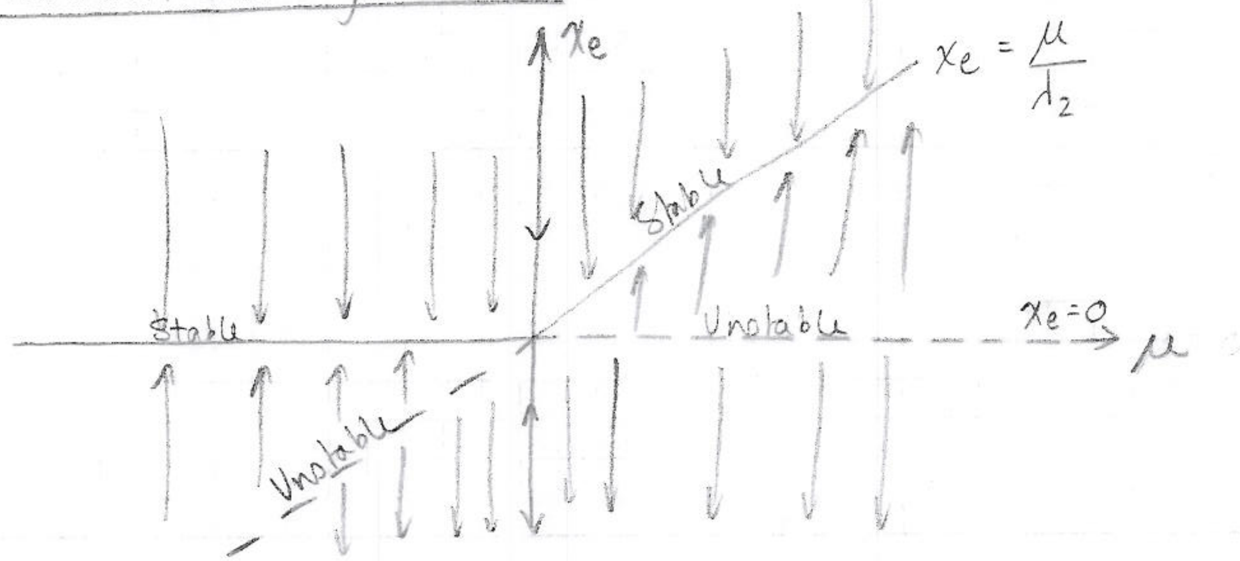
$\mu < 0$: Stable " " "

$\underline{x_e = \frac{\mu}{d_2}}$: $D_x f_\mu = \mu - 2 d_2 \frac{\mu}{d_2} = -\mu$

$\Rightarrow \mu > 0$: Stable equilibrium at $x = \frac{\mu}{d_2}$

$\mu < 0$: Unstable " " "

Transcritical bifurcation:



3. Note that $N(a)$ is positive real and so $-\frac{1}{N(a)}$ lies on the negative real axis.

Now $$G(jw) = \frac{(1-w^2-2jw)(4-w^2-4jw)}{(1+w^2)^2(4+w^2)^2}$$

$$Im(G(jw)) = \frac{-2w(6-3w^2)}{*} = 0$$

$$\Rightarrow w = \sqrt{2}$$

$\therefore$ $1 + N(a) \, Re(G(j\sqrt{2})) = 0$

check that $N(a) = 18$

Because $N(a)$ starts from b at $a=0$ and decreases after $a = \frac{1}{b}$, the equation $N(a) = 18$ has a solution if $b > 18$.

The frequency of oscillation will be close to $w = \sqrt{2}$.

4. $\dot{x}_1 = -x_1^3$, $\dot{x}_2 = x_1$ equilibria at $x_1 = 0$, $x_2 \in \mathbb{R}$

$$D_f\big|_{x_e} = \begin{bmatrix} -3x_1^2 & 0 \\ 1 & 0 \end{bmatrix}\Bigg|_{x_e} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\Longrightarrow$ Hartman-Grobman cannot be used

We can go back and look at the equations directly: The origin (or any point in $\mathbb{R}^2$) is <u>not</u> SISL for this system.

<u>Proof</u> Given $x_0 = (\delta, 0)$ for <u>any</u> $\delta > 0$ the ensuing $x_2(t; x_0, 0) \longrightarrow \infty$.

$\therefore$ $\forall \varepsilon$ $\cancel{\exists}\delta$ such that $\|x_0 - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \varepsilon$ $\forall t \geq 0$