1. $\dot{x}_1 = k_1 x_2 \left(1 - \frac{x_1}{1 + x_2^2}\right)$ $\quad k_1, k_2 > 0$

$\dot{x}_2 = k_2 - x_2 - \frac{4 x_1 x_2}{1 + x_2^2}$

(a) equilibria: $k_1 x_2 \left(1 - \frac{x_1}{1+x_2^2}\right) = 0 \quad (1)$

$k_2 - x_2 - \frac{4 x_1 x_2}{1 + x_2^2} = 0 \quad (2)$

from (1), $x_2 = 0$ or $\frac{x_1}{1+x_2^2} = 1$, but $x_2 = 0 \Rightarrow k_2 = 0$ $\therefore x_2 \neq 0$ @ equilibrium

$\therefore \frac{x_1}{1+x_2^2} = 1 \Rightarrow k_2 - x_2 - 4x_2 = 0$

$\Rightarrow x_2 = \frac{k_2}{5}$

$\therefore$ unique equilibrium @ $\left(1 + \left(\frac{k_2}{5}\right)^2, \frac{k_2}{5}\right)$

linearize: $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{-k_1 x_2}{(1+x_2^2)} & k_1 - \frac{k_1 x_1}{(1+x_2^2)} + \frac{2k_1 x_1 x_2^2}{(1+x_2^2)^2} \\ \frac{-4x_2}{(1+x_2^2)} & -1 - \frac{4x_1}{(1+x_2^2)} + \frac{8x_1 x_2^2}{(1+x_2^2)^2} \end{bmatrix}$

@ equilibrium, $\frac{x_1}{1+x_2^2} = 1$.

$\therefore \left.\frac{\partial f}{\partial x}\right|_{eq} = \frac{1}{1+x_2^2} \begin{bmatrix} -k_1 x_2 & 2k_1 x_2^2 \\ -4x_2 & -5 + 3x_2^2 \end{bmatrix}$

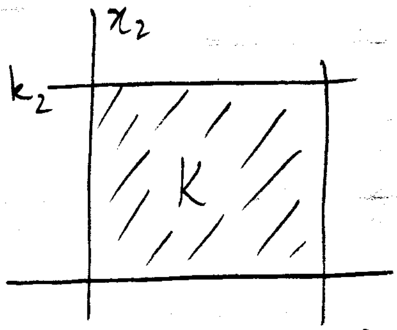
$\therefore$ characteristic equation:

$$s^2 + s(5 - 3x_2^2 + k_1 x_2) + 5k_1 x_2 = 0$$

@ equilibrium, $s^2 + s\left(5 - 3\left(\frac{k_2}{5}\right)^2 + \frac{k_1 k_2}{5}\right) + k_1 k_2 = 0 \quad (*)$

(b) Consider the region

$$K = \{(x_1,x_2) \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 \leq 1+k_2^2,\ x_2 \leq k_2\}$$



easy to show invariance of K:

on $x_1 = 0$, $\dot{x}_1 = k_1 x_2 \geq 0$ if $x_2 \geq 0$

on $x_2 = 0$, $\dot{x}_2 = k_2 > 0$

on $x_1 = 1+k_2^2$, $\dot{x}_1 = k_1 x_2 \left(1 - \frac{1+k_2^2}{1+x_2^2}\right) < 0$ if $0 < x_2 < k_2$

on $x_2 = k_2$, $\dot{x}_2 = -\frac{4x_1 k_2}{1+k_2^2} < 0$ for $x_1 > 0$

$\therefore$ K is invariant.

$\therefore$ in order to have (by Poincaré Bendixson) a closed orbit in K, we need that the only equilibrium inside K be unstable.

(note that the equilibrium @ $\left(1+\left(\frac{k_2}{5}\right)^2, \frac{k_2}{5}\right)$ is inside k).

examining the characteristic eqn (*), since $k_1 k_2 > 0$, a necessary & sufficient condition for the equilibrium to be unstable is that $5 - 3\left(\frac{k_2}{5}\right)^2 + \frac{k_1 k_2}{5} < 0$.

$\therefore$ By Poincaré-Bendixson, the system is guaranteed to have a periodic orbit in K if

$$5 - 3\left(\frac{k_2}{5}\right)^2 + \frac{k_1 k_2}{5} < 0$$

or $k_1 < \frac{3k_2}{5} - \frac{25}{k_2}$

(c) when $k_1 > \frac{3k_2}{5} - \frac{25}{k_2}$, equilibrium is stable

when $k_1 < \frac{3k_2}{5} - \frac{25}{k_2}$, equilibrium is unstable & is surrounded by an orbit

$\Rightarrow$ Supercritical Hopf bifurcation @ $k_1 = \frac{3k_2}{5} - \frac{25}{k_2}$

2 (a). $\dot{x}_1 = 1 - x_1 x_2^2$
$\dot{x}_2 = x_1$

(i) $\mathrm{div}(f) = -x_2^2$

(ii) equilibria do not exist ...

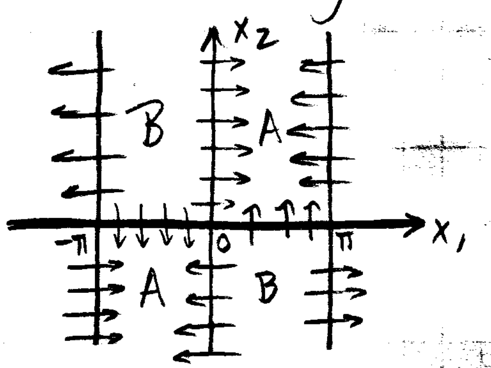
This system has no orbits, because, by <u>Index Theory</u>, an orbit must surround an equilibrium point. Thus, no equilibria $\Rightarrow$ no closed orbits.

(b) $\dot{x}_1 = x_2 \cos x_1$
$\dot{x}_2 = \sin x_1$

(Sol^n 1)
(i) simple solution: note that all equilibria are saddles, thus, by <u>index theory</u>, no closed orbits exist!

or, (Sol^n 2) (ii) $\mathrm{div}(f) = -x_2 \sin x_1$
now $\mathrm{div}(f) \equiv 0$ if $x_2 = 0$ or $x_1 = n\pi$.

$\therefore$ if we partition the state space into regions defined by $x_2 = 0$ and $x_1 = n\pi$, we know from <u>Bendixson's Thm</u> that an orbit cannot be completely enclosed in one of these regions:



on $x_1 = 0$, $\dot{x}_1 = x_2$, $\dot{x}_2 = 0$

on $x_2 = 0$, $\dot{x}_1 = 0$, $\dot{x}_2 = \sin x_1$

on $x_1 = \pi$, $\dot{x}_1 = -x_2$, $\dot{x}_2 = 0$

on $x_1 = -\pi$, $\dot{x}_1 = -x_2$, $\dot{x}_2 = 0$

ie regions A are positively invariant, regions B are negatively invariant:

$\therefore$ there are no closed orbits in this system, since if there were, they would have to leave a region A & enter a region B, which is impossible.

3. (a) The solution of the state equation

5

$$\dot{x}_1 = x_2 \qquad x_1(0) = x_{10}$$
$$\dot{x}_2 = k \qquad x_2(0) = x_{20}$$

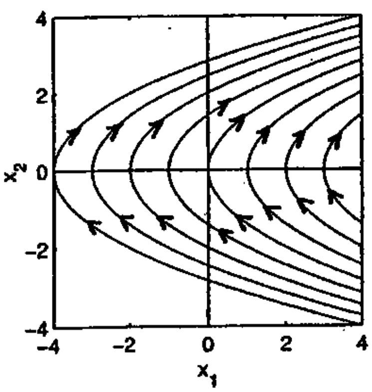
where $k = \pm 1$, is given by $x_2(t) = kt + x_{20}$

$$x_1(t) = \frac{1}{2}kt^2 + x_{20}t + x_{10}.$$

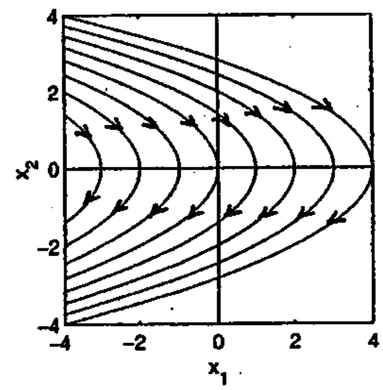
Eliminating t, we obtain:

$$x_1 = \frac{1}{2k}x_2^2 + c \qquad \left(c = x_{10} - \frac{x_{20}^2}{2k}\right)$$

Plotted below for different c:

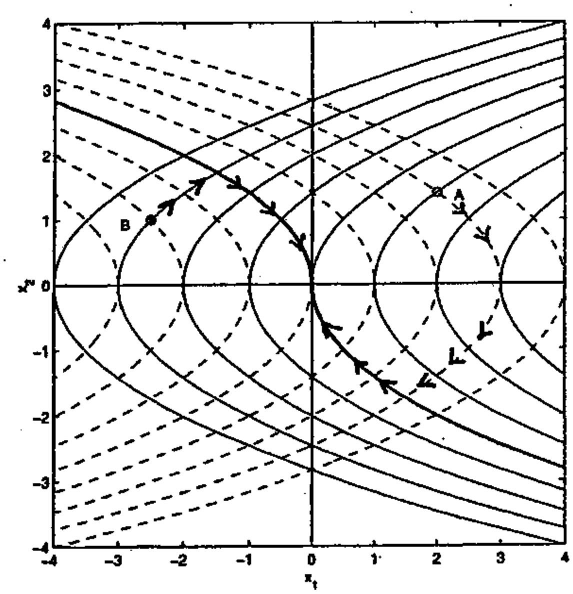


$u = 1$



$u = -1$

From the superimposed plot below, we see that trajectories can reach the origin only through 2 curves (highlighted)



∴ Switching policy:

① if start at pt to the right (ie. A) apply $u = -1$ until hit curve, then switch to $u = 1$

② B, apply $u = 1$ until hit curve, then apply $u = -1$.

<u>Important</u>: at origin? (apply $u = 0$)

3(b) $G(s) = \frac{1}{s^2} = \frac{\theta(s)}{u(s)}$

Plant equation $\quad \ddot{\theta} = u$

5

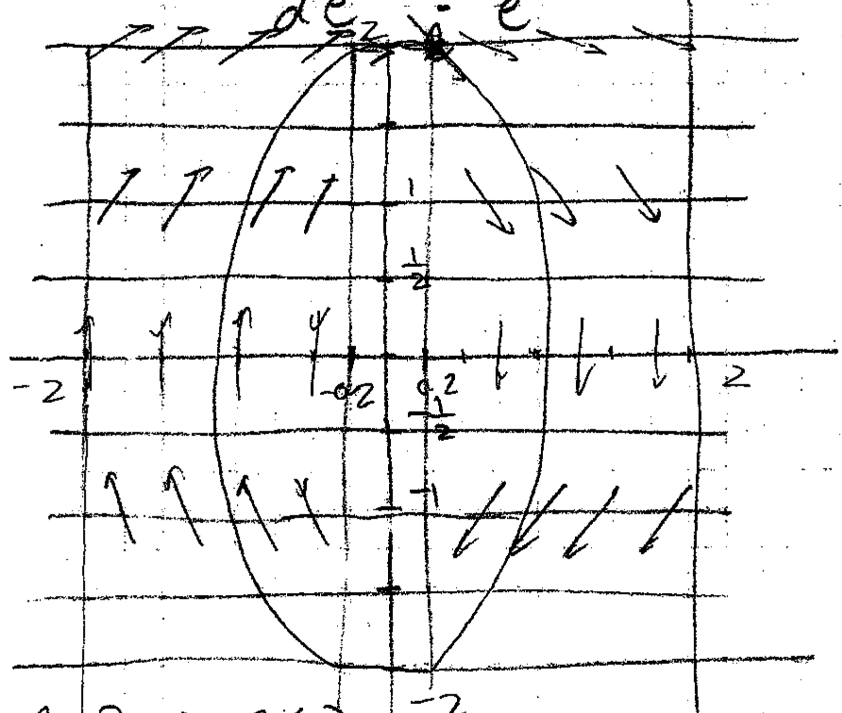
$$e(t) = r(t) - \theta(t)$$
$$\dot{e}(t) = -\dot{\theta}(t)$$
$$\ddot{e}(t) = -\ddot{\theta}(t)$$

$$\therefore \quad -\ddot{e} = n(e).$$

$$\Longrightarrow \quad \frac{d\dot{e}}{de}\dot{e} = -n(e) \Rightarrow \frac{d\dot{e}}{de} = \frac{-n(e)}{\dot{e}}$$

Three regions:

1. $e \leq -0.2 \Rightarrow \frac{d\dot{e}}{de} = \frac{1}{\dot{e}}$
2. $-0.2 < e < 0.2 \Longrightarrow \frac{d\dot{e}}{de} = 0$
3. $e \geq 0.2 \Longrightarrow \frac{d\dot{e}}{de} = -\frac{1}{\dot{e}}$



$e(0) = 0.2 \Rightarrow \theta(0) = 0.8$
$\dot{e}(0) = 2 \Rightarrow \dot{\theta}(0) = -2$

Claim $e(t) = \alpha + \beta \dot{e}(t)^2$ for some α, β

Proof $\dot{e}(t) = 2\beta\, \dot{e}(t)\, \ddot{e}(t)$

$$n(e) = 1 \Rightarrow \ddot{e} = \frac{1}{2\beta} \text{ if } \dot{e}(t) \neq 0$$

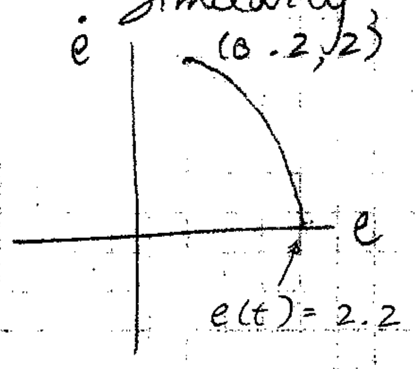
$$\Rightarrow \beta = -\frac{1}{2}$$

also, if $\dot{e} = 0$ then $e = 2.2 = \alpha$.

also, $e(0) = \alpha - \frac{1}{2}\dot{e}(0)^2 \Rightarrow \alpha = 2.2$

$$e(t) = 2.2 - \frac{1}{2}\dot{e}(t)^2$$

Similarly, for $n(e) = -1$, $e(t) = -1.8 + \frac{1}{2}\dot{e}(t)^2$



$$e(t) = 2.2 - \frac{1}{2}\dot{e}(t)^2$$

$$\Rightarrow \dot{e}(t) = \sqrt{4.4 - 2e(t)}$$

$$\Rightarrow \frac{de}{dt} = \sqrt{4.4 - 2e} \text{ for } \dot{e} > 0$$

$$\Rightarrow \frac{de}{\sqrt{4.4 - 2e}} = dt$$

$$\Rightarrow \frac{-2}{2}\sqrt{4.4 - 2e}\,\Big|_{0.2}^{2.2} = t$$

$$\Rightarrow t = -(0 - 2) = 2 \text{ seconds.}$$

Time to go from $(-0.2, 2)$ to $(0.2, 2)$

$$\ddot{e} = 0$$

$$\therefore \dot{e} = K = 2$$

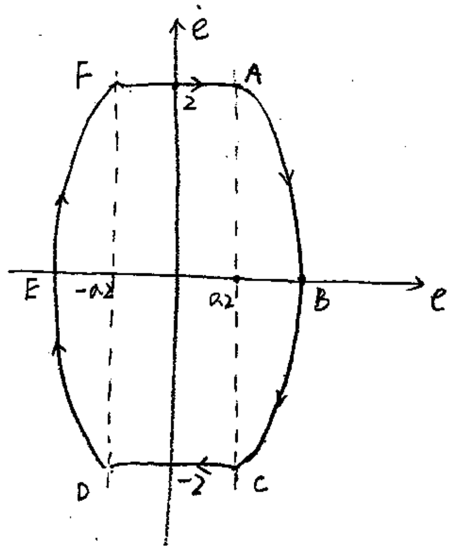
$$\Delta e = Kt \Rightarrow e = 2 \cdot t$$

$$\Rightarrow t = \frac{e}{2} = \frac{0.4}{2} = 0.2$$

$\Rightarrow$ Time for one period is:

$(4 \times 2) + 2(0.2) = \underline{\underline{8.4 \text{ seconds}}}$ $\leftarrow$

3.(b).



for $e > 0.2$

$$\ddot{e} = -n(e) = -1 \qquad \text{(constant)}$$

at A. $\dot{e} = 2$

at B. $\dot{e} = 0$

from $A \to B$, $\quad t_{A\to B} = \dfrac{\dot{e}_B - \dot{e}_A}{\ddot{e}} = \dfrac{0-2}{-1} = 2 \text{ sec.}$

from $F \to A$.

$$\dot{e} = 2. \quad \text{(constant)}$$

$$t_{F\to A} = \frac{e_A - e_F}{\dot{e}} = \frac{0.2-(-0.2)}{2} = 0.2 \text{ sec}.$$

symmetric, so period

$$T = 4 \cdot t_{A\to B} + 2\, t_{F\to A} = 4\times 2 + 2\times 0.2 = \underline{8.4 \text{ sec}}.$$

$$\dot{T} = -T + f(T)$$

4. (a) Heater on:

$$T(t) = e^{-t} \cdot T(t_{on}) + \int_{t_{on}}^{t} e^{-(t-\tau)} \cdot 100 \, d\tau$$

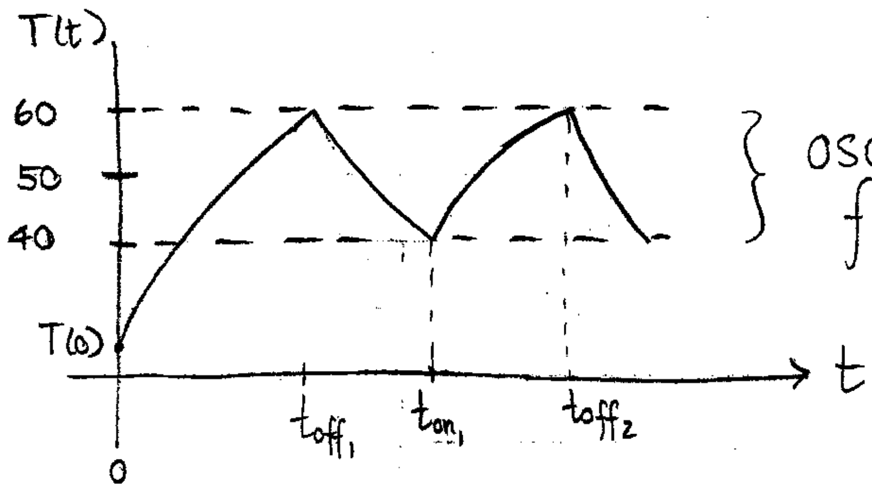
$$= T(t_{on}) e^{-t} + (1 - e^{t_{on}-t}) \cdot 100$$

↑ time @ which heater is switched on.

Heater off:

$$T(t) = T(t_{off}) e^{-t}$$

↑ time @ which heater is switched off



oscillates with frequency and amplitude indep. of T(0).

$\frac{1}{s+1} f(T) = T$

$\dot{T} + T$

b. $e = 50 - T$

$\therefore \dot{e} = -\dot{T}$ and thus $-\dot{e} + 50 - e = f(50-e)$

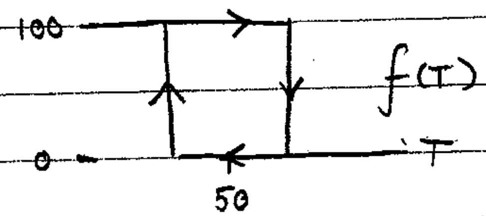
$$-(\dot{e} + e) = -50 + f(50-e)$$

$$\dot{e} + e = 50 - f(50-e)$$

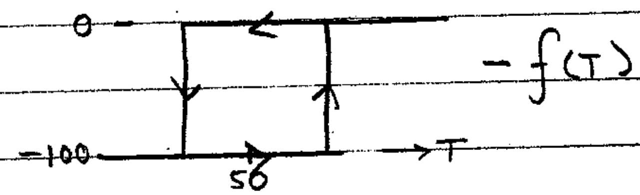
$\Rightarrow \dot{e} + e = n(-e)$

where $n(-e) = 50 - f(50-e)$

Note that



$\Downarrow$



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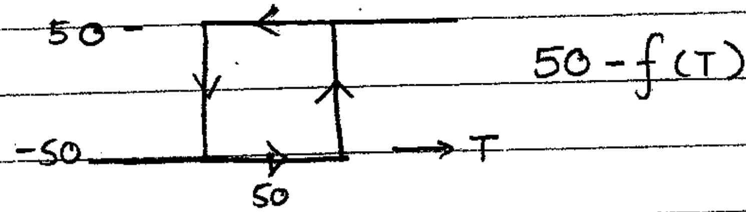
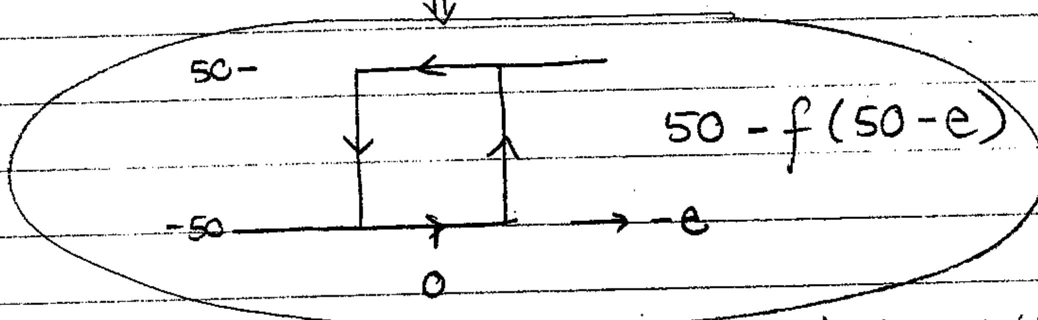
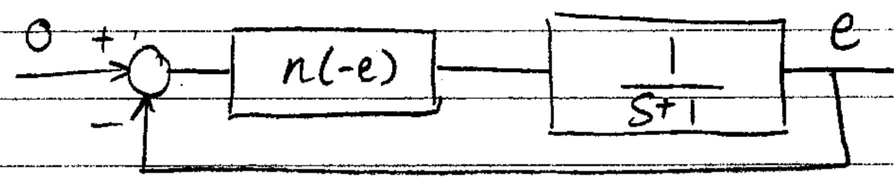


$\Downarrow$



← which is what we want!

Thus we rewrite the system as:



$$\Updownarrow$$

$$\dot{e} + e = n(-e)$$

We know from class that for this hysteresis element, the DF (for $a \geq \Delta$) is

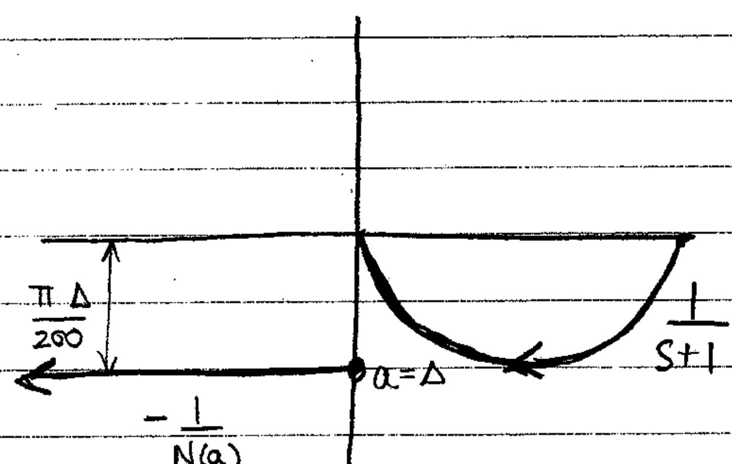
$$N(a) = \left(\frac{4}{a^2\pi}\sqrt{a^2-\Delta^2} - j\,\frac{4\Delta}{a^2\pi} \right) \cdot 50$$

from HW 5 problem

$$\therefore\ -\frac{1}{N(a)} = -\frac{a^2\pi}{200}\,\frac{1}{\sqrt{a^2-\Delta^2} - j\Delta} \cdot \frac{\sqrt{a^2-\Delta^2}+j\Delta}{\sqrt{a^2-\Delta^2}+j\Delta}$$

$$= -\frac{\pi}{200}\left(\sqrt{a^2-\Delta^2} + j\Delta \right)$$

doesn't depend on a!



$\Rightarrow$ Predicted no oscillation!

$\therefore$ DF prediction is wrong!
Why?