E209a: Analysis and Control of Nonlinear Systems
Problem Set 7 Solutions

1.) Phase-locked loop

$$\ddot{y}(t) + (a + b\cos(y(t)))\dot{y} + c \cdot \sin y(t) = 0$$

let $x_1 = y, \quad x_2 = \dot{y}$

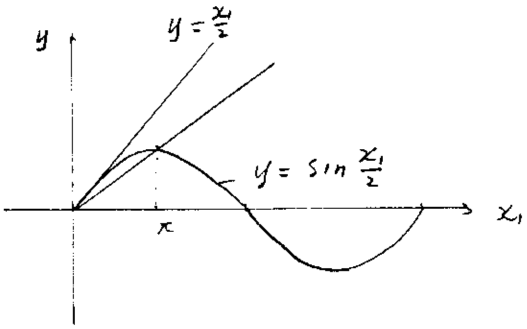
$$\Rightarrow \quad \dot{x}_1 = x_2$$
$$\dot{x}_2 = -(a + b\cos x_1) x_2 + c \sin x_1$$

$x_e = (0, 0)$ is an equilibrium

Lyapunov function candidate

$$V(x_1, x_2) = c(1$$

$$= c \cdot$$



let $G_r = \{x \in R^2, \|x\| \le \pi\}, \quad \Rightarrow \quad |x_1| < \pi$

from the plot above, we can see

$$\left|\frac{x_1}{\pi}\right| \le \left|\sin\frac{x_1}{2}\right| \le \left|\frac{x_1}{2}\right| \quad \text{on } G_r$$

$$\Rightarrow \quad 2c \cdot \left(\frac{x_1}{\pi}\right)^2 + \frac{x_2^2}{2} \le V(x_1, x_2) \le 2c \cdot \left(\frac{x_1}{2}\right)^2 + \frac{x_2^2}{2}$$

$$\Rightarrow \quad \min\left\{\frac{2c}{\pi^2}, \frac{1}{2}\right\} \|x\|^2 \le V(x_1, x_2) \le \max\left\{\frac{c}{2}, \frac{1}{2}\right\} \|x\|^2$$

$\Rightarrow$ V is PD and Decrescent on G_r. ①

$$\begin{aligned}\dot{V} &= c\cdot\sin x_1\cdot\dot{x}_1 + x_2\cdot\dot{x}_2\\ &= c\sin x_1\cdot x_2 + x_2\left(-c\sin x_1-(a+b\cos x_1)x_2\right)\\ &= -(a+b\cos x_1)x_2^2\end{aligned}$$

(a) if $a \geq b \geq 0$

$$a+b\cos x_1 \geq b(1+\cos x_1) \geq 0$$

$$\Rightarrow \quad \dot{V} \leq 0$$

V is a Lyapunov function. ②

From ① and ②

$\Rightarrow \quad x_e = (0, 0)$ is stable.

(b) if $a > b \geq 0$. use Lasalle's Theorem.

$$\Omega_c = \{x \in R^2,\ V(x) \leq c\}$$

from (a): Ω_c is bounded and $\dot{V}(x,t) \leq 0$.

$$\begin{aligned}S &= \{x \in \Omega_c,\ \dot{V} = 0\}\\ &= \{x\in\Omega_c,\ -(a+b\cos x_1)x_2^2 = 0\}\end{aligned}$$

$\because \quad a > b, \quad a+b\cos x_1 > 0$

$\therefore S = \{x \in \Omega_c,\ x_2 = 0\}$

on S, $\quad x_2 = 0, \quad \dot{x}_2 = 0 \quad \Rightarrow \quad \sin x_1 = 0$

$\Rightarrow \quad x_1 = 0 \quad$ (assume $\Omega_c \subset G_r$, hence exclude $x_1 = k\pi$)

$\Rightarrow \quad x_e = (0, 0)$ is asymptotically stable.

2.) $\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 + (1-x_1^2-x_2^2)x_2$

(a) $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ $\lambda = 0.5 \pm 0.866 i$ Unstable Focus

(b) We suspect limit cycle at $r=1$

=> Let $r^2 = x_1^2 + x_2^2$ $\theta = \tan^{-1}\frac{x_2}{x_1}$

=> $2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$ => $\dot{\theta} = \frac{1}{1+\frac{x_2^2}{x_1^2}} \times \frac{(-x_1^2 + x_1x_2(1-x_1^2-x_2^2) - x_2^2)}{x_1^2}$

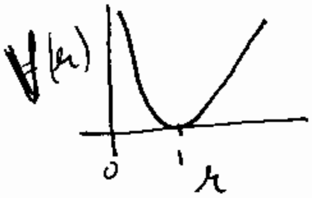
=> $r\dot{r} = x_2^2(1-x_1^2-x_2^2)$ $= -\frac{(x_1^2+x_2^2 - x_1x_2(1-x_1^2-x_2^2))}{x_1^2+x_2^2}$

=> $\dot{r} = x_2^2(1-r^2)/r$ $= -\frac{(r^2 - x_1x_2(1-r^2))}{r^2}$

=> When $r=1$

$\left.\begin{matrix} \dot{r} = 0 \\ \dot{\theta} = -1 \end{matrix}\right\}$ => Limit Cycle at $r=1$

(c) Choose Cake-pan shaped Lyapunov Function.



$V(r) = \frac{1}{r^2} + r^2 - 2$

=> $V(x) = \frac{1}{x_1^2+x_2^2} + x_1^2 + x_2^2 - 2$

$\dot{V}(x) = \left(2r - \frac{2}{r^3}\right)\dot{r} = \left(2r - \frac{2}{r^3}\right)x_2^2(1-r^2)/r$

$= 2x_2^2\frac{(r^4-1)(1-r^2)}{r^4} \le 0$ For $\forall x \in \mathbb{R}^2$

(Check this!)

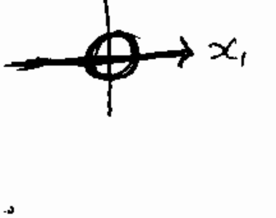
=> Lasalle's Principle:

Page 11-9

$\dot{V} \le 0$

$S = \{x : \dot{V}(x) = 0\}$

=> $S = \{x : x_2 = 0 \text{ or } r = 1\}$



Largest Invariant Subset of S.

If $x_2 = 0$ & $x_1 \neq \{0,1,-1\}$ => $x_2 \neq 0$ leaves S.

=> $M = \{x : r = 1\}$ or $x_1^2 + x_2^2 = 1$ → All trajectories converge to M. Not from (0,0) which is the limit cycle

3.) Satellite stabilization.

- States: $\begin{bmatrix} w \\ do \end{bmatrix} \in R^6$

: system equations:

$$I\dot{w} + w \times Iw = u$$
$$\dot{do} = -w \times do$$

with $u = -\alpha w + do \times bo$, $\alpha > 0$

- Find equilibria:

$\dot{w} = 0$ and $\dot{do} = 0$

$$\Rightarrow \begin{cases} w \times Iw = u = -\alpha w + do \times bo & \cdots \text{①} \\ w \times do = 0 & \cdots\cdots\cdots \text{②} \end{cases}$$

from ②, (suppose $w \neq 0$)

$do = k_1 w$, k_1 is a constant scalar.

plug into ①,

$w \times Iw - k_1(w \times bo) = -\alpha w$

the left hand side of the above equation is perpendicular to w, the right hand side is in the direction of w.

$\Rightarrow$ $w = 0$

Now from ①,

$do \times bo = 0 \Rightarrow do = k_2 bo$, k_2 is a constant.

since do and bo are unit vectors

$\Rightarrow$ $do = \pm bo$ (lined up).

equilibria:

$$\begin{bmatrix} 0 \\ bo \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -bo \end{bmatrix}$$

we now use Lyapunov theory to determine the stability of the equilibria.

$$V(w, do) = \frac{1}{2} w^T I w + \frac{1}{2} ||do + bo||^2$$

$$V(0, bo) = 2$$

$$V(0, -bo) = 0$$

since $I > 0$, $V(w, do)$ is PD.

$$\dot{V} = \frac{1}{2} \dot{w}^T I w + \frac{1}{2} w^T I \dot{w} + \frac{1}{2} (\dot{do} + \dot{bo})^T (do + bo) + \frac{1}{2} (do + bo)^T (\dot{do} + \dot{bo})$$

$$= \dot{w}^T I w + (\dot{do}^T + \dot{bo}^T)(do + bo) \quad \text{since } \dot{w}^T I w = w^T I \dot{w}$$

$\dot{bo} = 0$ because in the body frame, bo is fixed.

$$\Rightarrow \dot{V} = \underline{\dot{w}^T I} w + \dot{do}^T (do + bo) \qquad \text{(note: the underlined are equal \& } I^T = I\text{)}$$

$$= \underline{(-\alpha w + do \times bo - w \times I w)^T} \, w + (-w \times do)^T (do + bo)$$

$$= -\alpha w^T w + (do \times bo)^T \cdot w + (-w \times I w)^T \cdot w - (w \times do)^T do - (w \times do)^T bo$$

Using $F \cdot G \times H = G \cdot H \times F = H \cdot F \times G$

$$\therefore \ (do \times bo)^T w = (w \times do)^T bo$$

$$(-w \times I w)^T \cdot w = (w \times do)^T do = 0$$

$$\Rightarrow \dot{V} = -\alpha w^T \cdot w = -\alpha ||w||^2 \leq 0$$

system is SISL.

Applying Lasalle's theorem:

Looking at $x_e = \begin{bmatrix} 0 \\ -bo \end{bmatrix}$.

Choose $c_1 < 2$, and define

$$\Omega_{c_1} := \{ (w, d_0) \in R^6,\ \|d_0\| = 1 \mid V(w, d_0) \le c_1 \}$$

so $\begin{bmatrix} 0 \\ b_0 \end{bmatrix} \notin \Omega_{c_1}$.

Define $S \subset \Omega_c$ as

$$S := \{ (w, d_0) \in \Omega_{c_1} \mid \dot V(w, d_0) = 0 \}$$

$$= \{ (w, d_0) \in \Omega_{c_1} \mid -\partial \|w\|^2 = 0 \}$$

$$= \{ (w, d_0) \in \Omega_{c_1} \mid w = 0 \}$$

The largest invariant set in S is

$$M := \{ (w, d_0) \in S \mid w = 0 \text{ and } \underline{\underline{\dot w}} = 0 \}$$

on M, $w = 0 \Rightarrow \underline{\underline{\dot d_0}} = -w \times d_0 = 0$ (equilibria)

Since

$$M \subset S \subset \Omega_{c_1} \quad \text{and} \quad \begin{bmatrix} 0 \\ b_0 \end{bmatrix} \notin \Omega_{c_1}$$

$$\Rightarrow M = \{ (0, -b_0) \}$$

$\therefore$ By Lasalle's Theorem,

$(0, -b_0)$ is locally asymptotically stable.

$\therefore$ all initial conditions in Ω_{c_1} converge to $(0, -b_0)$.

4.) TRY $V(w) = \frac{1}{2}\left(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2\right)$

(a) LPDF ✓ as a lyapunov function candidate

$$\dot V(w) = J_1 w_1 \dot w_1 + J_2 w_2 \dot w_2 + J_3 w_3 \dot w_3$$
$$= 0$$

(i). $\Rightarrow$ origin is stable

it is <u>not</u> asymptotically stable — SHOW!

(ii) $u_i = -k_i w_i$

using same $V(w)$, we have

$$\dot V = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 < 0$$

$\therefore$ $-\dot V$ is PD

$\therefore$ origin is globally asymptotically stable

5.) (i) $\dot x_1 = x_2$, $\dot x_2 = -g(x_1) - h(x_1)x_2$

@ equilibrium,

$$x_2 = 0 \; ; \; g(x_1) + h(x_1)x_2 = 0$$
$$\Rightarrow x_2 = 0 \; ; \; g(x_1) = 0$$

if we assume that $g(x_1) = 0$ has an isolated root at the origin, then the origin is an isolated equilibrium point.

(ii) Take candidate lyapunov fn as, in class,

$$V(x) = \int_0^{x_1} g(y)\,dy + \frac{1}{2}x_2^2$$

$$\therefore \dot V(x) = g(x_1)x_2 - x_2 g(x_1) - h(x_1)x_2^2$$
$$= -h(x_1)x_2^2 \le 0$$

1b cont^{d}

now, $\boxed{\text{if } h(x_1) > 0}$ $\forall x_1 \in D$ (D contains origin)

Then $\dot{V} \le 0$ and

$$\begin{aligned} \dot{V} \equiv 0 \Rightarrow h(x_1)\, x_2^2 \equiv 0 &\Rightarrow x_2 \equiv 0 \\ &\Rightarrow g(x_1) = 0 \\ &\Rightarrow x_1 = 0 \end{aligned}$$

hence, by Lasalle, origin is asymp. stable.

Problem 6

The governing equation are:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - n(2x_1 + x_2)\end{aligned}$$

where

$$n(y) = \begin{cases} -1 & y < -1 \\ y & |y| \leq 1 \\ 1 & y > 1 \end{cases}$$

Let's try the Lyapunov candidate $V = \frac{1}{2}||x||^2$ which is by definition positive definite and decrescent. Taking the Lie derivative of V yields:

$$\dot{V}(x) = 2x_1x_2 - x_2n(2x_1 + x_2)$$

Now let's try to find a region in which we can guarantee that $V(x) \leq 0$. If $|2x_1 + x_2| \leq 1$, then

$$\dot{V}(x) = -x_2^2 \leq 0$$

Therefore, we can at least say that the equilibrium point is stable in the sense of Lyapunov, but since $\dot{V}(x)$ is not positive definite, let's try to use Lasalle's theorem to prove local asymptotic stability. First we need to define the Ω_c region in which $\dot{V}(x) \leq 0$.

$$\begin{aligned}|2x_1 + x_2| &\leq 1 \\ |2x_1 + x_2| &\leq 2||x|| + ||x|| = 3||x|| \leq 1\end{aligned}$$

Writing this in terms of our Lyapunov candidate we obtain

$$\Omega_c = \left\{ x \mid V \leq \frac{2}{9} \right\}$$

The subset of Ω_c in which $\dot{V}(x) = 0$ is

$$S = \left\{ x \mid \dot{V}(x) = 0 \right\}$$

which is $x_2 = 0$. The invariant subset of S is

$$\begin{aligned}x_1 - n(2x_1 + x_2) &= 0 \\ x_1 - n(2x_1) &= 0\end{aligned}$$

which can only be zero at $x_1 = 0$. Thus, the invariant subset of S is

$$M = \{x \mid x_1 = 0,\ x_2 = 0\}$$

Therefore, by Lasalle's theorem, the origin is locally asymptotically stable.