

E209A: Analysis and Control of Nonlinear Systems

Problem Set 6 Solutions

Michael Vitus
Gabe Hoffmann
Stanford University

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Problem 1

The governing equations are:

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 * x_2^2 \\ \dot{x}_2 &= -x_2 + x_2^3\end{aligned}\tag{1}$$

Using the function $V(x) = x_1^2 + x_2^2$ let's see what we can say about the stability of the origin. Since $V(x)$ is a class-K function, by definition $V(x)$ is a positive-definite and decrescent function. The Lie derivative is:

$$\dot{V} = 2(x_1^2 + x_2^2)(x_2^2 - 1)\tag{2}$$

Therefore, $\dot{V} \leq 0$ only when $x_2^2 \leq 1$. Therefore, the equilibrium is only locally stable on $G_r = \{x | x_2^2 \leq 1\}$.

Problem 2: Phase-Locked Loop.

We are to analyze the phase-locked loop described by

$$\ddot{y}(t) + (a + b \cos y(t))\dot{y} + c \sin y(t) = 0 \quad (3)$$

Let $x_1 = y$ and $x_2 = \dot{y}$. We need to show that $(0,0)$ is a stable equilibrium point if $a \geq b \geq 0$. The system is then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(a + b \cos x_1)x_2 - c \sin x_1 \end{aligned}$$

Given the Lyapunov function candidate $V(x_1, x_2) = c(1 - \cos x_1) + x_2^2/2$, we need to start by showing that $V(x, t)$ is positive definite and decrescent on G_r . We assume that $c > 0$ for the hint to be helpful. It is helpful to use the trig identity

$$\frac{1 - \cos x_1}{2} = \sin^2 \frac{x_1}{2} \quad (4)$$

Thus,

$$V = 2c \sin^2 \frac{x_1}{2} + \frac{x_2^2}{2} \quad (5)$$

For simplicity, choose $G_r = \{x \in \mathbb{R}^n : \|x\| < \pi\}$. Thus, V monotonically increases with $\|x\|$ from the origin until $x_1 = \pm\pi$, when $\sin^2 \frac{x_1}{2} = \sin^2 \frac{\pi}{2} = 1$, and the x_1 term is maximum with respect to x_1 . To show that the function is decrescent on G_r , we look at

$$\left| 2c \sin^2 \frac{x_1}{2} + \frac{x_2^2}{2} \right| \leq \|x\|^2 \max \left(\frac{c}{2}, \frac{1}{2} \right) \quad (6)$$

where the $\frac{c}{2}$ condition in the max function arises from

$$\left| \sin \frac{x_1}{2} \right| \leq \left| \frac{x_1}{2} \right| \quad \forall x_1 \quad (7)$$

and the $\frac{1}{2}$ condition in the max function is for the case where $x_2^2/2$ dominates if c is too small. Therefore, $\phi(\|x\|) = \max(\frac{c}{2}, \frac{1}{2})\|x\|^2$ is a class-K function that bounds V from above, proving that V is decrescent on G_r .

To show that V is positive definite on G_r , we use the fact that

$$\left| 2c \sin^2 \frac{x_1}{2} + \frac{x_2^2}{2} \right| \geq \min \left(\frac{2c}{\pi^2}, \frac{1}{2} \right) \|x\|^2 \text{ when } \|x\| \leq 1 \quad (8)$$

where the $\frac{1}{2}$ condition in the min function is from the case where $x_2^2/2$ dominates if c is too big. The $\frac{2c}{\pi^2}$ term is from solving for the coefficient necessary for x_2^2 to intercept $2c \sin^2 \frac{x_1}{2}$ at $x_1 = \pi$. That is, when $2c \sin^2 \frac{\pi}{2} = k\pi^2$. The parabola can only intercept the square of the sin at one point between 0 and π , and it does so from below, so the condition in (8) is proven true. Thus, $\min(\frac{2c}{\pi^2}, \frac{1}{2})\|x\|^2$, another class-K function, bounds V from below on G_r , proving that V is positive definite on G_r .

Now, we need to show that $-\dot{V}$ is positive definite.

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 \\ &= \left(2c \sin \frac{x_1}{2} \cos \frac{x_1}{2} \right) (x_2) + (x_2) (-(a + b \cos x_1)x_2 - c \sin x_1) \\ &= cx_2 \sin x_1 - x_2^2(a + b \cos x_1) - c \sin x_1 \\ &= -x_2^2(a + b \cos x_1) \end{aligned}$$

Clearly, $-x_2^2$ is always ≤ 0 . The term $(a + b \cos x_1)$ is always ≥ 0 , because $a \geq b \geq 0$, so $a + b \cos x_1 \geq b(1 + \cos x_1) \geq 0$. Therefore, $\dot{V} \leq 0$, so the origin is a stable equilibrium point.

An alternative method to prove stability, although it goes against the implication of the hint, is to find the Jacobian linearization of the system about the equilibrium.

$$Df|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ bx_2 \sin x_1 - c \cos x_1 & -(a + b \cos x_1) \end{array} \right] \Big|_{(0,0)} = \left[\begin{array}{cc} 0 & 1 \\ -c & -(a + b) \end{array} \right] \quad (9)$$

This leads to a characteristic equation of $\lambda^2 + (a + b)\lambda + c = 0$. Solving for λ , we see that $\lambda = \frac{1}{2} \left(-(a + b) \pm \sqrt{(a + b)^2 - 4c} \right)$. Thus, the real part of the eigenvalue is always less than zero, by the triangle inequality.

There is the rather large exception to this stability claim, $a = b = 0$, which is allowed by the constraints, where the roots are purely imaginary. By Hartman-Grobman, we cannot tell the stability of the system from the linearization in this case. We can, however, tell from the Lyapunov analysis that we already did. If $a = b = 0$, then $\dot{V} = 0$, so the solution trajectories would follow level curves of V encircling the origin.

Problem 3.

$$\begin{aligned}\ddot{y} &= u \\ u &= -k(y)\end{aligned}$$

Using $V(y, \dot{y}) = \frac{1}{2}\dot{y}^2 + \int_0^y k(\sigma)d\sigma$ show that the origin is stable in the sense of Lyapunov if $k(0) = 0$ and $\frac{dK}{dy}(0) > 0$. Let $x_1 = y$ and $x_2 = \dot{y}$.

$$\dot{V}(x) = x_2\dot{x}_2 + k(x_1)\dot{x}_1 \tag{10}$$

$$\dot{V}(x) = x_2(-k(x_1)) + k(x_1)x_2 \tag{11}$$

$$\dot{V}(x) = 0 \tag{12}$$

$$\tag{13}$$

Therefore, $V(x) \leq 0$. Setting $\dot{x}_1 = \dot{x}_2 = 0$, yields $(k^{-1}(0), 0)$ and since $k(0) = 0$, the origin is an equilibrium point.

Now we need to show that $V(x)$ is positive-definite and decrescent. Since $V(x)$ is time-invariant, it is decrescent. Since $\frac{dK}{dx_1}(0) > 0$, there is no deadband in $k(x_1)$. Therefore, at least locally around $(0, 0)$ we can bound the function $V(x)$ from below.

Therefore, $V(x)$ is LPDF, decrescent Lyapunov function.

Therefore, the origin is locally stable.

Problem 4: Control.

We are to consider the system

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 - x_2 - x_1x_3 + u \\ \dot{x}_3 &= x_1 + x_1x_2 - 2ax_3\end{aligned}$$

where $a > 0$ and $b > 0$ are constants. We are asked if using Lyapunov theory, we can design a control law $u(x)$ which globally stabilizes the origin. Let's choose a positive definite, decrescent candidate Lyapunov function, $V(x)$, and see if we can define $u(x)$ such that $\dot{V}(x) \leq 0$. Let's try

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 \quad (14)$$

Then, we find the Lie derivative.

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 + x_3\dot{x}_3 \\ &= x_1a(x_2 - x_1) + x_2(bx_1 - x_2 - x_1x_3 + u) + x_3(x_1 + x_1x_2 - 2ax_3) \\ &= ax_1x_2 - ax_1^2 + bx_1x_2 - bx_2^2 - x_1x_2x_3 + x_2u + x_1x_3 + x_1x_2x_3 - 2ax_3^2 \\ &= (a+b)x_1x_2 + x_1x_3 - ax_1^2 - bx_2^2 - 2ax_3^2 + x_2u\end{aligned}$$

In order to eliminate the terms that can be positive in the derivative, we could chose

$$u(x) = -x_1(a+b) - \frac{x_1x_3}{x_2} \quad (15)$$

However, this control law goes to infinity when $x_2 \rightarrow 0$. Let's see what the real impact of the x_1x_3 term is. Perhaps it is cancelled by the quadratic terms. Using the control law $u(x) = -x_1(a+b)$, we would have

$$\dot{V}(x) = x_1x_3 - ax_1^2 - bx_2^2 - 2ax_3^2 \quad (16)$$

Clearly, the x_2 term is less than or equal to zero. The remainder of the terms can be simplified by seeing that we can rewrite x_1x_3 in a quadratic form.

$$-(x_1 - cx_3)^2 = -x_1^2 + 2cx_1x_3 - c^2x_3^2 \Rightarrow x_1x_3 = \frac{1}{2c}x_1^2 + \frac{c}{2}x_3^2 - (x_1 - cx_3)^2 \quad (17)$$

where c is an arbitrary scaling constant. Substituting (17) back into (16) we obtain

$$\dot{V}(x) = -\frac{1}{2}(x_1 - x_3)^2 - \left(a - \frac{1}{2c}\right)x_1^2 - \left(2a - \frac{c}{2}\right)x_3^2 \quad (18)$$

Therefore, if $a \geq \frac{1}{2c}$ and $a \geq \frac{c}{2}$, then $\dot{V} \leq 0$, and the origin is globally stabilized. To find the value of c that constrains a the least, we solve for c at the point where both inequalities transition from true to false at the same time. That leads to the system of equations

$$\begin{aligned}a &= \frac{1}{2c} \\ 2a &= \frac{c}{2}\end{aligned}$$

The solution is $c = \sqrt{2}$ and $a = \frac{1}{2\sqrt{2}}$. Thus, if $a \geq \frac{1}{2\sqrt{2}}$, then the Lie derivative is less than or equal to zero, and the origin is globally stabilized. However, we do not have the option of picking a , therefore this candidate function does not work. Therefore, let's try a different candidate function.

$$V(x) = \frac{1}{2}(\alpha x_1^2 + \beta x_2^2 + \delta x_3^2) \quad (19)$$

As long as α, β , and $\delta > 0$, the function $V(x)$ is positive definite and decrescent. Taking the Lie derivative yields,

$$\begin{aligned} \dot{V} &= \alpha x_1 \dot{x}_1 + \beta x_2 \dot{x}_2 + \delta x_3 \dot{x}_3 \\ \dot{V} &= -a\alpha x_1^2 - \beta x_2^2 - 2a\delta x_3^2 + (\delta - \beta)x_1 x_2 x_3 + \delta x_1 x_3 + (b\beta + a\alpha)x_1 x_2 + \beta u \end{aligned}$$

Let's choose $\beta = \delta$, which will eliminate the term $(\delta - \beta)x_1 x_2 x_3$. Also, let's choose $u = -\frac{1}{\beta}(b\beta + a\alpha)x_1$, which will eliminate one of the two positive terms. Substituting and simplifying yields,

$$\dot{V} = -a\alpha x_1^2 - \beta x_2^2 - 2a\beta x_3^2 + \beta x_1 x_3$$

Now we need to show that $\dot{V} \leq 0$ for some α and β . \dot{V} can be written in the form

$$\dot{V} = -x^T Q x \quad (20)$$

where

$$Q = \begin{bmatrix} a\alpha & 0 & -0.5\beta \\ 0 & \beta & 0 \\ -0.5\beta & 0 & 2a\beta \end{bmatrix} \quad (21)$$

We now need to show that Q is positive-semidefinite so that $V(x) \leq 0$. In order for $Q \geq 0$, the diagonal of Q needs to be positive, and the determinant needs to be greater than or equal 0. Therefore, $a > 0$, $\alpha > 0$, and $\beta > 0$. The determinant of Q is

$$-(2a^2\alpha\beta - 0.25\beta^2) \quad (22)$$

Setting the determinant equal to zero, and solving for β yields

$$\beta = 8a^2\alpha \quad (23)$$

Choosing $\alpha = 1$, yields $\beta = 8a^2$. Therefore, with the function $V(x) = \frac{1}{2}x_1^2 + 4a^2(x_2^2 + x_3^2)$ which is positive definite and decrescent, and with $u = -(b + \frac{1}{8a})x_1$, we have shown that $\dot{V}(x) \leq 0$. Therefore, we have found a control input which makes the origin stable in the sense of Lyapunov.

Problem 5: RLC circuit with passive nonlinear resistor.

The state equations of this RLC circuit are

$$\dot{x} = y - f(x) \tag{24}$$

$$\dot{y} = -x \tag{25}$$

where x is the current, y is the capacitor voltage, and $f(x)$ is the nonlinear resistor's effect. We can assume that $f(x)$ is a continuous function, because a resistor is a physical device. We are to show that if the resistor is strictly passive, then the equations admit only one stable equilibrium. Let's start by finding the equilibria.

$$\dot{y} = -x = 0 \Rightarrow x = 0 \tag{26}$$

$$\dot{x} = y - f(x) = 0 \Rightarrow y = f(x) = f(0) \tag{27}$$

So the equilibrium is at $(0, f(0))$. It is unique, because $xf(x) = 0$ only when $x = 0$. Next, we use the stored energy function for the circuit, $W = (x^2 + y^2)/2$, as the Lyapunov function. It is obviously positive definite and decrescent. Then,

$$\begin{aligned} \dot{V} &= \dot{W} \\ &= x\dot{x} + y\dot{y} \\ &= x(y - f(x)) + y(-x) \\ &= -xf(x) \end{aligned}$$

The resistor is passive, so $xf(x) = 0$ only at $x = 0$, and is positive otherwise. Therefore, $\dot{V} \leq 0$, so the minimum of W is a stable equilibrium point. There is only one equilibrium point, so the one equilibrium point is stable.

Problem 6.

We consider the system

$$\dot{x}_1 = -x_2 + \epsilon x_1(x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \quad (28)$$

$$\dot{x}_2 = x_1 + \epsilon x_2(x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \quad (29)$$

We are to show that the linearization is inconclusive to determine the stability of the origin, and use the method of Lyapunov to study the stability of the origin between $\epsilon = 1$ and $\epsilon = -1$.

First, notice that the origin is clearly an equilibrium point. The Jacobian linearization of the system is

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad (30)$$

where

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 2\epsilon x_1^2(x_1^2 + x_2^2) \cos(x_1^2 + x_2^2) + \epsilon(3x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \\ \frac{\partial f_1}{\partial x_2} &= 2\epsilon x_1 x_2(x_1^2 + x_2^2) \cos(x_1^2 + x_2^2) + 2\epsilon x_1 x_2 \sin(x_1^2 + x_2^2) - 1 \\ \frac{\partial f_2}{\partial x_1} &= 2\epsilon x_1 x_2(x_1^2 + x_2^2) \cos(x_1^2 + x_2^2) + 2\epsilon x_1 x_2 \sin(x_1^2 + x_2^2) + 1 \\ \frac{\partial f_2}{\partial x_2} &= 2\epsilon x_2^2(x_1^2 + x_2^2) \cos(x_1^2 + x_2^2) + \epsilon(x_1^2 + 3x_2^2) \sin(x_1^2 + x_2^2) \end{aligned}$$

Evaluating the Jacobian linearization at the origin, it is simply

$$Df|_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad (31)$$

Thus, the eigenvalues of the linearized system are simply $\lambda = \pm j$. Because the eigenvalues are purely imaginary, according to Hartman-Grobman, we cannot determine the stability of this equilibrium.

Let us pick the simple and common Lyapunov function, $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, and see how much information it gives us. The Lie derivative is

$$\begin{aligned} \dot{V} &= x_1 \dot{x}_1 + x_2 \dot{x}_2 \\ &= x_1(-x_2 + \epsilon x_1(x_1^2 + x_2^2) \sin(x_1^2 + x_2^2)) + x_2(x_1 + \epsilon x_2(x_1^2 + x_2^2) \sin(x_1^2 + x_2^2)) \\ &= \epsilon(x_1^2 + x_2^2)^2 \sin(x_1^2 + x_2^2) \end{aligned}$$

The $(x_1^2 + x_2^2)^2$ term is always ≥ 0 . Around the origin, the $\sin(x_1^2 + x_2^2)$ term is ≥ 0 for $\|x\| < \sqrt{\pi}$. So, let us take $G_r = \{x \in \mathbb{R}^n : \|x\| < \sqrt{\pi}\}$. Then, in G_r , $\dot{V} > 0$ if $\epsilon > 0$, and $\dot{V} < 0$ if $\epsilon < 0$. Therefore, if $\epsilon < 0$, the origin is stable, and if $\epsilon > 0$, the origin is unstable. If $\epsilon = 0$, then clearly the system is a continuum of circular orbits, as can be seen by looking at the system equations.

Problem 7: Proving Stability of a Boundary control scheme.

The governing equation is

$$m \frac{\partial^2}{\partial t^2} W(x, t) - T \frac{\partial^2}{\partial x^2} W(x, t) = 0 \quad (32)$$

A vertical force u is applied at the free end of the string, at $x = 1$. The balance of forces in the vertical direction yields

$$u(t) = T \left. \frac{\partial W(x, t)}{\partial x} \right|_{x=1} \quad (33)$$

for all $t \geq 0$. We are to show that if the boundary control

$$u(t) = -k \left. \frac{\partial W(x, t)}{\partial t} \right|_{x=1} \quad (34)$$

is applied, where $k > 0$, then $\dot{E}(t) \leq 0$ for all $t \geq 0$. We are given that

$$E(t) = \frac{1}{2} \int_0^1 m \left(\frac{\partial W(x, t)}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^1 T \left(\frac{\partial W(x, t)}{\partial x} \right)^2 dx \quad (35)$$

where the first term is the kinetic energy, and the second term is the strain energy. Taking the derivative, we obtain

$$\dot{E}(t) = \int_0^1 m \left(\frac{\partial W}{\partial t} \right) \left(\frac{\partial^2 W}{\partial t^2} \right) dx + \int_0^1 T \left(\frac{\partial W}{\partial x} \right) \left(\frac{\partial^2 W}{\partial x \partial t} \right) dx \quad (36)$$

Using (32), we see that

$$\frac{\partial^2 W}{\partial t^2} = \frac{T}{m} \frac{\partial^2 W}{\partial x^2} \quad (37)$$

Thus,

$$\dot{E}(t) = T \left(\int_0^1 \left(\frac{\partial W}{\partial t} \right) \left(\frac{\partial^2 W}{\partial x^2} \right) dx + \int_0^1 \left(\frac{\partial W}{\partial x} \right) \left(\frac{\partial^2 W}{\partial x \partial t} \right) dx \right) \quad (38)$$

Now, we need to simplify this, so we use integration by parts, as given by the hint. To do so, the equations need to be of the form $\int_a^b u dv = uv|_a^b - \int_a^b v du$. We apply this form to the second integral above, by absorbing the second order derivatives into the differential term, and taking $u = \frac{\partial W}{\partial x}$ and $v = \frac{\partial W}{\partial t}$. Then,

$$\begin{aligned} \dot{E}(t) &= T \left(\int_0^1 \left(\frac{\partial W}{\partial t} \right) d \left(\frac{\partial W}{\partial x} \right) + \int_0^1 \left(\frac{\partial W}{\partial x} \right) d \left(\frac{\partial W}{\partial t} \right) \right) \\ &= T \left(\int_0^1 \left(\frac{\partial W}{\partial t} \right) d \left(\frac{\partial W}{\partial x} \right) + \left. \frac{\partial W}{\partial x} \frac{\partial W}{\partial t} \right|_0^1 - \int_0^1 \left(\frac{\partial W}{\partial t} \right) d \left(\frac{\partial W}{\partial x} \right) \right) \\ &= T \left. \frac{\partial W}{\partial x} \frac{\partial W}{\partial t} \right|_0^1 \\ &= T \left. \frac{\partial W}{\partial x} \frac{\partial W}{\partial t} \right|_{x=1} - T \left. \frac{\partial W}{\partial x} \frac{\partial W}{\partial t} \right|_{x=0} \end{aligned}$$

The first line was found by using $\frac{\partial}{\partial x} \frac{\partial W}{\partial x} = \frac{\partial^2 W}{\partial x^2} \rightarrow d \left(\frac{\partial W}{\partial x} \right) = \frac{\partial^2 W}{\partial x^2} dx$ for the first integral, and similar calculus for the second integral. The integration limits throughout the problem are from $x = 0$ to $x = 1$. The second line was found by applying integration by parts to the second integral in line 1. The third line was found by realizing that the two integrals in line two are equal and have opposite signs.

We use $W(0, t) = 0$ to see that $\frac{\partial W}{\partial t}\big|_{x=0} = 0$. In other words, the left end of the string is fixed in place. This eliminates the second term in the equation above, yielding

$$\begin{aligned}\dot{E}(t) &= \left(T \frac{\partial W}{\partial x}\bigg|_{x=1}\right) \frac{\partial W}{\partial t}\bigg|_{x=1} \\ \dot{E}(t) &= u(t) \frac{\partial W}{\partial t}\bigg|_{x=1}\end{aligned}$$

where the last step used (33). Now, if we define $u(t)$ as in (34), with $k > 0$, we have

$$\dot{E}(t) = -k \left(\frac{\partial W}{\partial t}\right)_{x=1}^2 \leq 0 \quad \forall t \geq 0 \tag{39}$$