E209A: Analysis and Control of Nonlinear Systems
Problem Set 4 Solutions

Michael Vitus
Stanford University

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Problem 1

Part a

The governing equations are:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \nu_1 x + \nu_2 y + x^2 - x^2 y \end{aligned} \tag{1}$$

Setting $\dot{x} = 0$ and $\dot{y} = 0$ and solving for the equilibria yields:

$$\boxed{(0,0)} \quad \boxed{(\sqrt{-\nu_1}, 0)} \quad \boxed{(-\sqrt{-\nu_1}, 0)} \tag{2}$$

Therefore there are 3 equilibria for $\nu_1 < 0$ and 1 equilibrium for $\nu_1 \geq 0$. Consequently, there is a pitchfork bifurcation at $\nu_1 = 0 \; \forall \; \nu_2$.

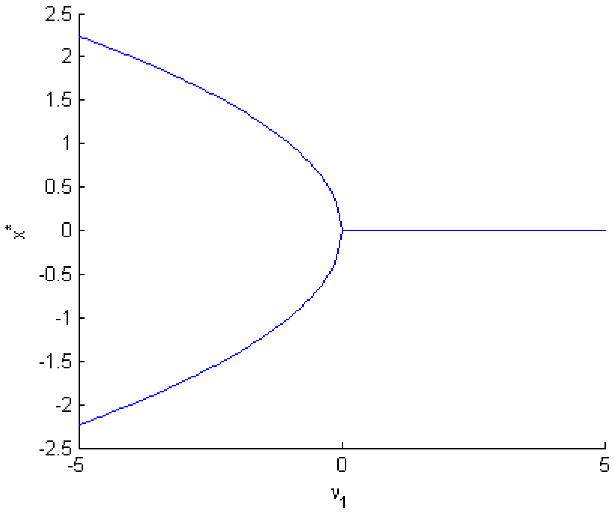

Figure 1: Problem 1 - Plot of x^* as a function of ν_1.

Part b

First looking at Bendixson's theorem,

$$div(f) = \nu_2 - x^2 \tag{3}$$

Therefore, there cannot be any limit cycles for $\nu_2 < 0$ since Eqn. (3) doesn't change sign or equals zero.

Let's try to rule out more regions by checking the equilibria. The equilibrium at $(0,0)$ has eigenvalues of

$$\lambda = \frac{1}{2}\left(\nu_2 \pm \sqrt{\nu_2^2 + 4\nu_1}\right) \tag{4}$$

For $\nu_1 > 0$ the equilibrium at $(0,0)$ is a saddle. Thus, for $\nu_1 > 0$, the only equilibrium is a saddle, and we know the index of a saddle is -1. Therefore, there are no limit cycles for $\nu_1 > 0$.

The regions in which we can rule out limit cycles is:

$$\boxed{\nu_2 < 0 \text{ or } \nu_1 > 0} \tag{5}$$

Problem 2 - Unfolding of a double zero eigenvalue

The governing equations are:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \mu_1 + \mu_2 y + x^2 + xy \end{aligned} \tag{6}$$

Solving for the equilibrium points

$$\boxed{\left(\pm\sqrt{-\mu_1}, 0\right)} \tag{7}$$

Even though mathematically, there are equilibrium points defined for $\mu_1 > 0$, they are not valid in a real system. Therefore, the equilibrium points are only valid for $\mu_1 \leq 0$. To determine the stability of the equilibrium points, we will need to linearize the state model about them. The Jacobian is

$$Df = \begin{bmatrix} 0 & 1 \\ 2x + y & \mu_2 + x \end{bmatrix} \tag{8}$$

Let's first evaluate the stability of the equilibrium point at $(\sqrt{-\mu_1}, 0)$.

$$Df = \begin{bmatrix} 0 & 1 \\ 2\sqrt{-\mu_1} & \mu_2 + \sqrt{-\mu_1} \end{bmatrix} \tag{9}$$

Solving for the characteristic equation

$$\lambda^2 - \left(\mu_2 + \sqrt{-\mu_1}\right)\lambda - 2\sqrt{-\mu_1} = 0 \tag{10}$$

which has roots at

$$\lambda = \frac{1}{2}\left(\mu_2 + \sqrt{-\mu_1} \pm \sqrt{\left(\mu_2 + \sqrt{-\mu_1}\right)^2 + 8\sqrt{-\mu_1}}\right) \tag{11}$$

Therefore, $\forall \mu_2,\ \mu_1 < 0$, the equilibrium point at $(\sqrt{-\mu_1}, 0)$ will be a saddle point.

Now, let's analyze the stability of the equilibrium point at $(-\sqrt{-\mu_1}, 0)$.

$$Df = \begin{bmatrix} 0 & 1 \\ -2\sqrt{-\mu_1} & \mu_2 - \sqrt{-\mu_1} \end{bmatrix} \tag{12}$$

Solving for the characteristic equation

$$\lambda^2 - \left(\mu_2 - \sqrt{-\mu_1}\right)\lambda + 2\sqrt{-\mu_1} = 0 \tag{13}$$

which has roots at

$$\lambda = \frac{1}{2}\left(\mu_2 - \sqrt{-\mu_1} \pm \sqrt{\left(\mu_2 - \sqrt{-\mu_1}\right)^2 - 8\sqrt{-\mu_1}}\right) \tag{14}$$

Examining Eqn. (14) we see that for

- $\mu_2 > \sqrt{-\mu_1}$: the equilibrium point is unstable
- $\mu_2 < \sqrt{-\mu_1}$: the equilibrium point is stable
- $\mu_2 = \sqrt{-\mu_1}$: based on the Jacobian linearization we cannot be sure what type of equilibrium point we have since the eigenvalues are on the $j\omega$-axis, which is from the Hartman-Grobman theorem.

From this we can see that the line defined by $\mu_2 = \sqrt{-\mu_1}$ with $\mu_1 < 0$ is an interesting area to look at in more detail. After simulating the system around this line, we can verify that for a constant $\mu_2 > 0$, as you start to decrease μ_1 from zero, the equilibrium point at $(-\sqrt{-\mu_1}, 0)$, transforms from unstable to stable with an unstable periodic orbit around it. By definition, this is the subcritical Hopf bifurcation. There are also saddle-node bifurcations which occur on $\mu_1 = 0$, $\mu_2 \neq 0$. Figure 2 shows representative phase portraits in the (μ_1, μ_2) plane.

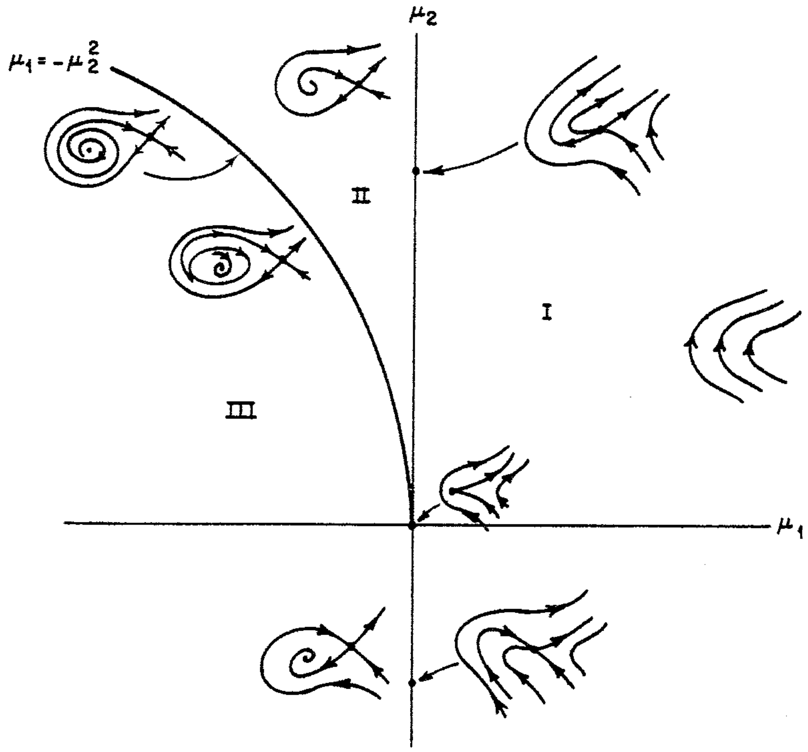

Figure 2: Problem 2 - Phase portrait in the (μ_1, μ_2) plane

Problem 3 - A Bifurcation study of a 3-D Pendulum

The governing equations are:

$$ml\ddot{\theta} = ml\omega^2 \cos\theta \sin\theta - mg\sin\theta \tag{15}$$

Letting $x_1 = \theta$ and $x_2 = \dot{theta}$ yields

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \left(\omega^2 \cos x_1 - \tfrac{g}{L}\right)\sin x_1 \end{aligned} \tag{16}$$

The equilibrium points are

$$\begin{aligned} &\boxed{(k\pi, 0)} \\ &\boxed{(\cos^{-1}\frac{g}{\omega^2 L}, 0)} \quad \text{if } \tfrac{g}{\omega^2 L} \leq 1 \end{aligned} \tag{17}$$

Therefore we have a critical angular speed at

$$\omega_o = \sqrt{fracgL} \tag{18}$$

To determine the stability of the equilibrium points, we will need to linearize the state model about them. The Jacobian is

$$Df = \begin{bmatrix} 0 & 1 \\ \omega^2 \cos(2x_1) - \frac{g}{L}\cos x_1 & 0 \end{bmatrix} \tag{19}$$

$\omega < \omega_o$

Therefore $\frac{g}{\omega^2 L} > 1$ and $\cos^{-1}\frac{g}{\omega^2 L}$ doesn't exist. Thus there is only one equilibrium point at $(0,0)$. To determine the stability, we need to examine the Jacobian linearization at this point.

$$Df|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \omega^2 - \frac{g}{L} & 0 \end{bmatrix} \tag{20}$$

The characteristic equation is

$$\lambda^2 + (\frac{g}{L} - \omega^2) = 0 \tag{21}$$

which has eigenvalues at

$$\lambda = \pm\sqrt{\omega_o^2 - \omega^2}j \tag{22}$$

Since the eigenvalues are on the jw-axis, we cannot determine the stability based upon the linearization.

Constructing a first integral yields,

$$\begin{aligned} \frac{\partial H}{\partial x_1} &= -\left(\omega^2 \cos x_1 - \tfrac{g}{L}\right)\sin x_1 \\ \frac{\partial H}{\partial x_2} &= x_2 \end{aligned} \tag{23}$$

After intregration

$$H = \frac{1}{2}x_2^2 + \frac{1}{4}\omega^2\cos(2x_1) - \omega_o^2\cos x_1 \tag{24}$$

Since H is a first integral, H = const, will give us a trajectory of the system. Evaluating Eqn. (24) at (ϵ_1, ϵ_2) as $\epsilon_1, \epsilon_2 \to 0$ yields

$$H = \frac{1}{2}\epsilon_2^2 + \frac{1}{4}\omega^2(1 - \frac{(2\epsilon_1)^2}{2!}) - \omega_o^2(1 - \frac{\epsilon_1^2}{2!}) = const \tag{25}$$

Simplifying,

$$\frac{1}{2}\epsilon_2^2 + \frac{1}{2}(\omega_o^2 - \omega^2)\epsilon_1^2 = const \tag{26}$$

Since $\omega < \omega_o$, we obtain ellipses. Therefore, the equilibrium point at $(0,0)$ is a center.

$\omega = \omega_o = \sqrt{\frac{g}{L}}$

To determine the stability, we need to examine the Jacobian linearization at this point.

$$Df|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{27}$$

which has two zero eigenvalues. Therefore, $\mu = \omega - \omega_o = 0$ is a bifurcation point at $(0,0)$.

$\omega > \omega_o$

There are two equilibria at

$$\boxed{(k\pi, 0)} \quad \boxed{(\cos^{-1}\frac{g}{\omega^2 L}, 0)} \tag{28}$$

To determine the stability at $(0,0)$, we need to examine the Jacobian linearization at this point.

$$Df|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \omega^2 - \omega_o^2 & 0 \end{bmatrix} \tag{29}$$

which has eigenvalues of $\lambda = \pm\sqrt{\omega^2 - \omega_o^2}$. Therefore equilibrium point at $(0,0)$ is a saddle.

For the equlibrium point at $(\cos^{-1}(\frac{g}{\omega^2 L}), 0)$

$$Df|_{(\cos^{-1}(\frac{g}{\omega^2 L}),0)} = \begin{bmatrix} 0 & 1 \\ \frac{\omega_o^4 - \omega^4}{\omega^2} & 0 \end{bmatrix} \tag{30}$$

which has eigenvalues of $\lambda = \pm\frac{1}{\omega}\sqrt{\omega^4 - \omega_o^4}j$. Since the eigenvalues are on the jw-axis, we cannot determine the stability based upon the linearization. Evaluating Eqn. (24) at $(\cos^{-1}(\frac{g}{\omega^2 L}) + \epsilon_1, \epsilon_2)$ as $\epsilon_1, \epsilon_2 \to 0$ will also show that the level sets of H are closed orbits around $(\cos^{-1}(\frac{\omega_o^2}{\omega^2}), 0)$. Therefore it is a center.

Figure 3 is a sketch of the bifurcation plot in (x_1, x_2, μ) for the bifurcation parameter $mu = \omega - \omega_o$.

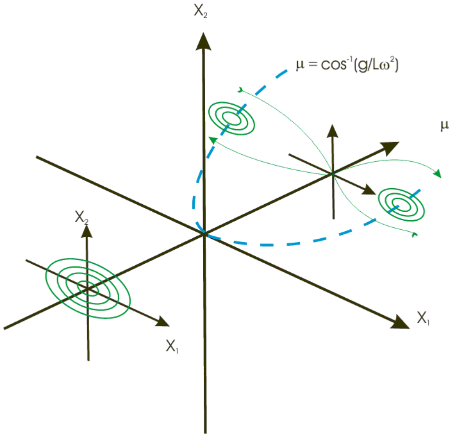

Figure 3: Problem 3 - Bifurcation Sketch.

Problem 4 - Van der Pol Oscillator

The governing equations are:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1 + \epsilon\left(x_2 - x_2^3\right) \end{array} \tag{31}$$

Part a - Equilibrium Points

Setting $\dot{x}_1 = \dot{x}_2 = 0$ and solving for x_1 and x_2 yields the only equilibrium point as

$$\boxed{(0,0)} \tag{32}$$

To determine the stability, we need to examine the Jacobian linearization at this point.

$$Df = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon(1-3x_2^2) \end{bmatrix} \tag{33}$$

Evaluating it at the equilibrium point

$$Df|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon \end{bmatrix} \tag{34}$$

The characteristic equation is

$$\lambda^2 - \epsilon\lambda + 1 = 0 \tag{35}$$

The roots of the characteristic equation are

$$\lambda = \frac{1}{2}(\epsilon \pm \sqrt{\epsilon^2 - 4}) \tag{36}$$

The type and stability of the equilibrium point will depend on ϵ. The following are the different types of equilibrium point that are achievable

$$\begin{array}{ll} 0 < \epsilon < 2 & \text{unstable focus} \\ \epsilon = 2 & \text{improper unstable node} \\ \epsilon > 2 & \text{unstable node} \end{array} \tag{37}$$

Part b - $\epsilon = 1$

Show that the region given is positively invariant.

AB

$$\dot{x}_1 = x_2 \geq 0 \tag{38}$$

Therefore the trajectories point inward

BC

This line segment is a circular arc centered at $(0,0)$. Transforming the system into polar coordinates

$$\begin{array}{rcl} r & = & (x_1^2 + x_2^2)^{\frac{1}{2}} \\ \theta & = & \arctan\frac{x_2}{x_1} \end{array} \tag{39}$$

After differentiating r and simplifying we obtain

$$\begin{array}{rcl} \dot{r} & = & \frac{1}{r}(x_1\dot{x}_1 + x_2\dot{x}_2) \\ \dot{r} & = & \frac{1}{r}(x_1x_2 + x_2(-x_1 + x_2 - x_2^3) \\ \dot{r} & = & \frac{1}{r}x_2^2(1 - x_2^2) \end{array} \tag{40}$$

From B←C, $x_2 > 1$, therefore the trajectories point inward.

CD

$$\dot{x}_2 = -x_1 + x_2 - x_1^3 \tag{41}$$

Therefore CD is above the isocline $x_1 = x_2 - x_2^3$, and $x_2 \geq 1$. Hence, $\dot{x}_2 < 0$, which shows that the trajectories enter the region.

DE

The same analysis on BC.

EF

Define the outward pointing normal to the line segment as

$$u = [\frac{1}{a-b} \;\; 1] \tag{42}$$

To guarantee that the trajectories enter the region, we need the dot product between the outward pointing normal and the system's dynamics to be less than or equal to zero. On EF, we know that $x_1 = (b-a)x_2 + a$ with $0 \leq x_2 \leq 1$. The dot product is

$$u \cdot \dot{x} = \frac{x_2}{a-b} + (-x_1 + x_2 - x_2^2) \tag{43}$$

which we need to show is less than or equal to zero. Let the Eqn. (43) be defined as $g(x)$. Differentiating $g(x)$ yields

$$\begin{array}{rcl} \dot{g}(x) & = & -2(a-b)x_2 + 1 + (a-b)(a-b+1) \\ & \geq & -2(a-b)1 + (a-b)^2 + (a-b) + 1 \\ & = & (a-b)^2 - (a-b) + 1 \\ & = & (a-b-\frac{1}{2})^2 + \frac{3}{4} > 0 \end{array} \tag{44}$$

which means that $g(x)$ is monotonely increasing on $0 \leq x_2 \leq 1$. Therefore, if we can show that $g(1) \leq 0$, we can guarantee that $g(x_2) \leq 0$ on EF. Therefore if a and b satisfy the following condition then the trajectories will enter the region EF.

$$\begin{array}{l} g(1) = 1 + (a-b)(-b) \leq 0 \\ \boxed{b^2 - ab + 1 \leq 0} \end{array} \tag{45}$$

From symmetry, we can conclude the region is positively invariant.

Now we can modify our region from the inside to exclude a unit circle centered at $(0,0)$ whose equation satisfies

$$x_2^2 \leq 1 \tag{46}$$

Therefore, transforming the system into polar coordinates and evaluating $\dot{r}$ we obtain

$$\dot{r} = \frac{1}{r} x_2^2 (1 - x_2^2) \tag{47}$$

which is always greater than zero. Therefore, we have constructed a positively invariant region with no equilibrium points. Therefore, by Poincare'-Bendixson's Theorem, there must be a limit cycle inside.

Problem 5

There are two stable linear systems.

$$\dot{x} = A_1 x \qquad \dot{x} = A_2 x \tag{48}$$

where

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix} \tag{49}$$

There are two switching policies. For switching policy (i), $\dot{x} = A_1 x$ is valid when $x_1 x_2 < 0$ and $\dot{x} = A_2 x$ is valid when $x_1 x_2 \geq 0$. For switching policy (ii), $\dot{x} = A_1 x$ is valid when $x_1 x_2 \geq 0$ and $\dot{x} = A_2 x$ is valid when $x_1 x_2 < 0$.

Part a

To determine the stability of the two policies, we need to look at the interaction at the switching lines.

- For the switching line at $x_1 = 0$

$$\frac{dx_2}{dx_1}|_{A_1} = \frac{-1}{10} \qquad \frac{dx_2}{dx_1}|_{A_2} = \frac{-1}{100} \tag{50}$$

- For the switching line at $x_2 = 0$

$$\frac{dx_2}{dx_1}|_{A_1} = \frac{100}{1} \qquad \frac{dx_2}{dx_1}|_{A_2} = \frac{10}{1} \tag{51}$$

Switching policy (i)

Figure 4 shows an illustration of the slopes at the switching lines, which illustrates that the angle of the slope vector entering the switching line is larger than the angle of the slope vector exiting it. Therefore, we can conclude that the trajectories are going to spiral outward from $(0, 0)$, and that the system is un stable.

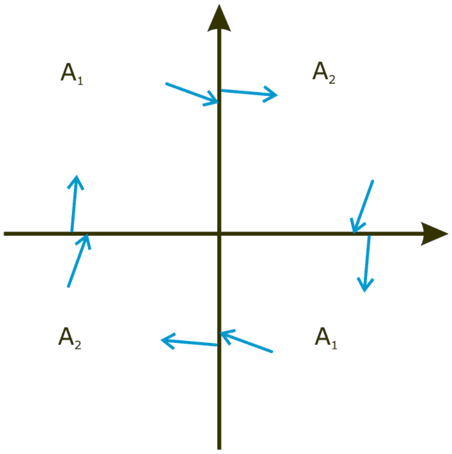

Figure 4: Problem 5 - switching policy (i)

Switching policy (ii)

Figure 5 shows an illustration of the slopes at the switching lines, which illustrates that the angle of the slope vector entering the switching line is smaller than the angle of the slope vector exiting it. Therefore, we can conclude that the trajectories are going to spiral inward toward $(0, 0)$, and that the system is stable.

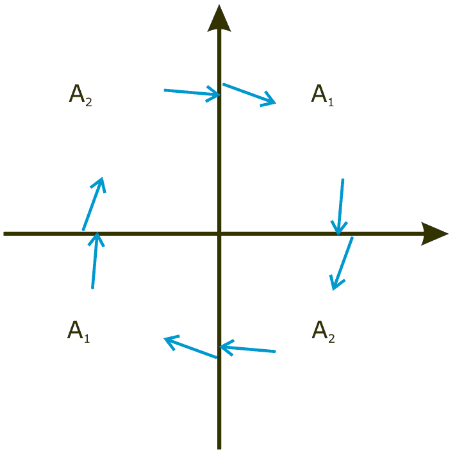

Figure 5: Problem 5 - switching policy (ii)

Part b

By considering the how the system behaves along the level curves of the function

$$V(x) = 2.75x_1^2 + 0.275x_2^2 \tag{52}$$

we can prove the stability of both systems.

System 1

Differentiating Eqn. (52) and plugging in the system dynamics results in

$$\dot{V}(x) = -5.5x_1^2 - 0.55x_2^2 \tag{53}$$

Therefore, $\dot{V}(x) < 0$ for all x_1 and x_2 not equal to zero, which depicts the trajectories going toward the equilibrium point at $(0, 0)$

System 2

Differentiating Eqn. (52) and plugging in the system dynamics results in

$$\dot{V}(x) = -5.5x_1^2 + 544.5x_1x_2 - 0.55x_2^2 \tag{54}$$

Therefore, $\dot{V}(x) < 0$ only when $x_1x_2 < 0$ which is in quadrants 2 and 4.

Therefore, from analyzing how the switched systems behave along the level curves of Eqn. (52) we have determined that the switching policy (ii) is the only stable one.

Figures 6 and 7 show the two systems.

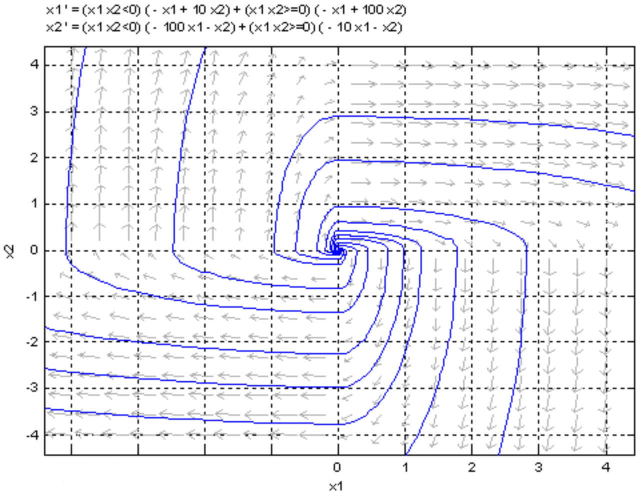

Figure 6: Problem 5 - switching policy (i)

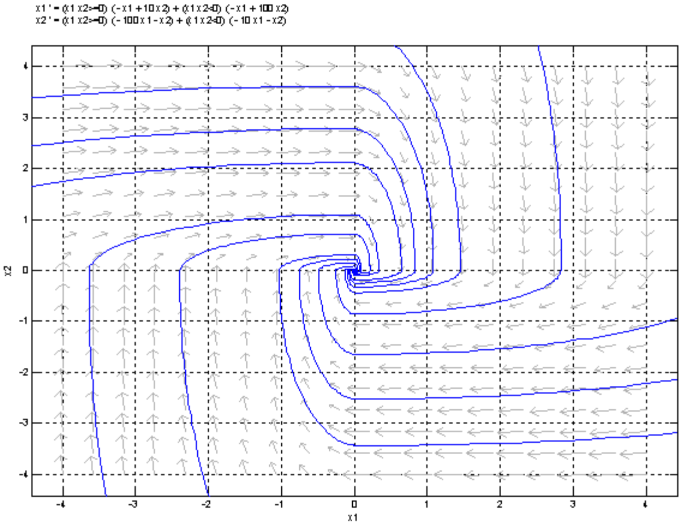

Figure 7: Problem 5 - switching policy (ii)