E209A Analysis and Control of Nonlinear Systems
Problem Set 8

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Issued 3/9; Due when you pick up your final.

Note: You should finish this problem set before taking the final exam, as there will be some problems on the final similar to those below. Solution sets for this problem set will be available after Wednesday March 14, once you hand in your solutions.

Problem 1: Domains of Attraction. Consider the damped nonlinear oscillator

$$\ddot{y} + 2\zeta\dot{y} + (1 - y)y = 0 \tag{1}$$

where ζ is a constant, with $0 < \zeta < 1$.

Using the state variable definition, $x_1 = y$, $x_2 = \frac{\dot{y}+\zeta y}{\gamma}$, where $\gamma = \sqrt{1-\zeta^2}$, find an estimate of the domain of attraction of the equilibrium at the origin $(x_1, x_2) = (0, 0)$, using the indirect method of Lyapunov.

Problem 2. Consider the system

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 - x_2 - x_1x_3 + u \\ \dot{x}_3 &= x_1 + x_1x_2 - 2ax_3 \end{aligned}$$

Where $a > 0$ and $b > 0$ are constants. Design a control law $u(x)$ which globally stabilizes the origin.

Problem 3: Regular Pendulum.

Consider the pendulum equation:

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT \tag{2}$$

where $a > 0$, $b \geq 0$, $c > 0$, and θ is the angle that the rod makes with the vertical axis, T is the torque applied to the pendulum. We will assume that the torque is the control input. Suppose we would like to stabilize the pendulum at an angle $\theta = \delta$. For the pendulum to maintain equilibrium at $\theta = \delta$, the torque must have a steady state component T_{ss} which satisfies $0 = -a\sin\delta + cT_{ss}$. Choose as state variables $x_1 = \theta - \delta$ and $x_2 = \dot{\theta}$, and the control as $u = T - T_{ss}$. Assume $a = c = 10, \delta = \pi/4$, and $b = 0$.

(a) Using Jacobian Linearization, linearize the system about the origin. Now, using linear state feedback with gains $K = [k_1 \ k_2]$ with $k_1 = 2.5$ and $k_2 = 1$ around this linear system, show that the resulting closed loop system is locally asymptotically stable.

(b) Find a Lyapunov function for the closed loop system in (a), and use it to estimate the region of attraction.

(c) Now, design a stabilizing state feedback control law using feedback linearization (using output as $y = \theta$), locating the closed loop eigenvalues at the same locations as used in part (a) above. How would you compare the performance of this closed loop system with that computed in (a) above?

(d) (most interesting part) Finally, design a controller to move the pendulum from its stable equilibrium at $(0, 0)$ to a new equilibrium point at $(\pi/2, 0)$. To do this, choose a *desired output trajectory* $y_D(t)$, which is the output of a second order transfer function:

$$\frac{1}{(\tau s+1)^2} \tag{3}$$

driven by a step input. The choice of time constant τ determines the speed of motion from the initial to the final position. However, this speed is limited by the motor driving the pendulum: there is a maximum control input T that this motor can supply.

Consider nominal parameters $a = c = 10$ and $b = 1$, an outer loop, linear (PD) control law with proportional constant $K_p = 400$, derivative constant $K_d = 20$, and constraints $|T| \leq 2$. Consider values of τ ranging from $\tau = 0.05$ to $\tau = 0.25$, and design a nonlinear control law which achieves good tracking of the reference signal.

Problem 4: Dynamic Extension.

The following is a simple kinematic model of an automobile.

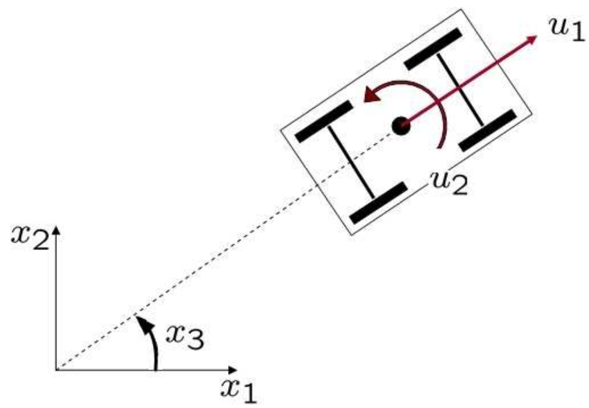

Figure 1: A kinematic model of an automobile.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cos x_3 & 0 \\ \sin x_3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5}$$

where x_1 and x_2 represent the position of the automobile in Cartesian coordinates, x_3 is the heading angle, and u_1 and u_2 represent the speed and the rate of change of heading angle (steering rate) respectively.

(a) What is the vector relative degree of the system? Is this system feedback linearizable?

(b) Consider a new state variable, $z_1 = u_1$, and a new input $w_1 = \dot{z}_1$. Under what condition does the above system augmented with this new state variable (and input) have well defined vector relative degree? Design a stabilizing input-output feedback linearizing control law to track a desired trajectory $(x_{1_d}(t), x_{2_d}(t))$.

(c) What are the internal dynamics of the corresponding closed-loop system obtained in part (b)?

Problem 5: Feedback Linearization for Robot Control. One of the great successes of feedback linearization is in robot control (so much so that roboticists use their own term "computed torque control" for feedback linearization). Consider the single link manipulator with flexible joint shown in Figure 2 below. The joint flexibility is modelled by the spring with spring constant k; q_1 and q_2 represent the joint angles at the base and at the manipulator respectively, u represents an input torque at the base, J is the moment of inertia of the base joint, I is the moment of inertia of the end effector, M is the mass of the end effector.

The equations of motion are:

$$J\ddot{q}_1 - k(q_2 - q_1) = u \tag{6}$$

$$I\ddot{q}_2 + MgL\sin(q_2) + k(q_2 - q_1) = 0 \tag{7}$$

Define state vector as $x = [q_1, \dot{q}_1, q_2, \dot{q}_2]^T$. Consider the SISO control problems of having EITHER $y = q_1$ OR $y = q_2$ as output. In each case, perform feedback linearization and determine the zero dynamics, and plot the

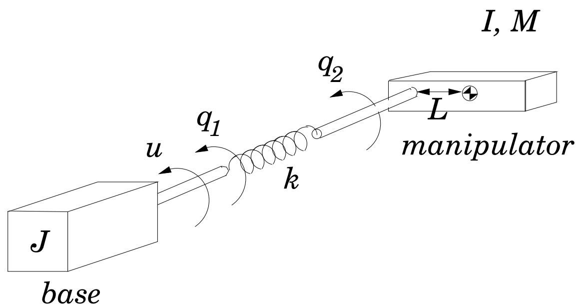

Figure 2: Single link manipulator with flexible joint.

zero dynamics for $I = J = M = L = k = 1$ and for $g = 9.8$. Which output would you prefer to control and why?