E209A Analysis and Control of Nonlinear Systems
Problem Set 7

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Problem 1: Phase-Locked Loop (Problem 2 from Problem Set 6).

Reconsider the phase-locked loop of Problem 2 in Problem Set 6. You showed that $(0,0)$ is a stable equilibrium point if $a \geq b \geq 0$. Now, show that $(0,0)$ is an asymptotically stable equilibrium point if $a > b \geq 0$.

Problem 2.

Consider the second order system

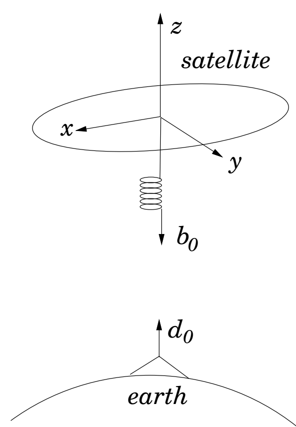
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (1 - x_1^2 - x_2^2)x_2 \end{aligned} \tag{1}$$

(a) Discuss the stability of the origin.

(b) Find the (only) limit cycle for this system.

(c) Prove using a suitable Lyapunov function and Lasalle's principle that all trajectories not starting from the origin converge to the limit cycle.

Problem 3: Satellite Stabilization.



Consider the satellite rigid body as shown with its body frame centered on its center of mass and axes aligned with the principal axes of inertia. Let

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} \tag{2}$$

be the moment of inertia of the satellite, where $I_1 > 0$, $I_2 > 0$, and $I_3 > 0$.

Let $b_0 \in \Re^3$ be a unit vector representing the direction of an antenna on the satellite, and let $d_0(t) \in \Re^3$ be a unit vector representing the direction of the ground antenna with respect to the satellite body coordinates.

Euler's equations of motion say that if $\omega \in \Re^3$ is the vector of angular velocities of the satellite about the principal axes (in body coordinates) and $u \in \Re^3$ is the external torque applied by the thrusters on the satellite, then the dynamics of the satellite are given by

$$I\dot{\omega} + \omega \times I\omega = u \tag{3}$$

Here, $\times$ refers to the cross product in $\Re^3$.

The aim of this problem is to find a control law u to align b_0 with $-d_0$. In the body frame, d_0 is not fixed, rather it rotates with angular velocity $-\omega$, so that

$$\dot{d}_0 = -\omega \times d_0 \tag{4}$$

Now, consider the control law

$$u = -\alpha\omega + d_0 \times b_0 \tag{5}$$

applied to the satellite/antenna system (equations (3),(4)), where $\alpha \in \Re$, $\alpha > 0$. I would like you to show that this control law makes $d_0(t)$ line up with $-b_0$.

In your analysis, use as state vector:

$$\begin{bmatrix} \omega \\ d_0 \end{bmatrix} \in \Re^6 \tag{6}$$

What is (are) the equilibrium point(s) of the resulting system in $\Re^6$? Now use the Lyapunov function candidate

$$V(\omega, d_0) = \frac{1}{2}\omega^T I\omega + \frac{1}{2}||d_0 + b_0||^2 \tag{7}$$

to show that control law (5) makes $d_0(t)$ line up with $-b_0$ for a given set of initial conditions. HINT: the cross product between two vectors $x, y \in \Re^3$ is

$$x \times y = \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ -x_2y_1 + x_1y_2 \end{bmatrix} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} y \tag{8}$$

Problem 4.

(a) Euler equations for a rotating rigid spacecraft are given by:

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3 \end{aligned} \tag{9}$$

where ω_1 to ω_3 are the components of the angular velocity vector ω along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moments of inertia.

(i) Show that with $u_i = 0$ for $i = 1, 2, 3$ the origin $\omega = 0$ is stable. Is it asymptotically stable?

(ii) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$ where k_i are positive constants. Show that the origin of the closed loop system is globally asmptotically stable.

Problem 5. Consider the equation:

$$\ddot{y} + h(y)\dot{y} + g(y) = 0 \tag{10}$$

where g and h are differentiable, with continuous derivatives.

(i) Find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(ii) Now, find further conditions on g and h to ensure that the origin is asymptotically stable.

Problem 6. Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - n(2x_1 + x_2)\end{aligned} \tag{11}$$

where $n(\cdot)$ represents the *saturation function*:

$$n(y) = \begin{cases} -1, & y < -1 \\ y, & |y| \leq 1 \\ 1, & y > 1 \end{cases} \tag{12}$$

What is the most you can say about the Lyapunov stability of the origin?