

# E209A Analysis and Control of Nonlinear Systems

## Problem Set 6

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**Problem 1.** Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2^2 \\ \dot{x}_2 &= -x_2 + x_2^3\end{aligned}$$

Using the function  $V = x_1^2 + x_2^2$ , what does the Lyapunov Stability Theorem say about the stability of the equilibrium at the origin?

**Problem 2: Phase-Locked Loop.** A phase-locked loop in communication networks can be described by

$$\ddot{y}(t) + (a + b \cos y(t))\dot{y} + c \sin y(t) = 0 \quad (1)$$

Show that  $(0, 0)$  is a stable equilibrium point if  $a \geq b \geq 0$ . (HINT: Consider the Lyapunov function candidate  $V(x_1, x_2) = c(1 - \cos x_1) + x_2^2/2$ , where  $x_1 = y$ , and  $x_2 = \dot{y}$ ).

**Problem 3.** Consider a double integrator with nonlinear output feedback

$$\begin{aligned}\ddot{y} &= u \\ u &= -k(y)\end{aligned}$$

Use the function  $V(y, \dot{y}) = \dot{y}^2/2 + \int_0^y k(\sigma)d\sigma$  to show that if  $k(0) = 0$  and  $\frac{\partial k}{\partial y}(0) > 0$  then the origin is a locally stable equilibrium.

**Problem 4: Control.** Consider the system

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= bx_1 - x_2 - x_1 x_3 + u \\ \dot{x}_3 &= x_1 + x_1 x_2 - 2ax_3\end{aligned}$$

Where  $a > 0$  and  $b > 0$  are constants. Using Lyapunov theory, can you design a control law  $u(x)$  which globally stabilizes the origin?

**Problem 5: RLC circuit with passive nonlinear resistor.** Consider the series RLC circuit shown in Figure 1, with  $L = C = 1$  and a nonlinear resistor with voltage-current characteristic given by a nonlinear

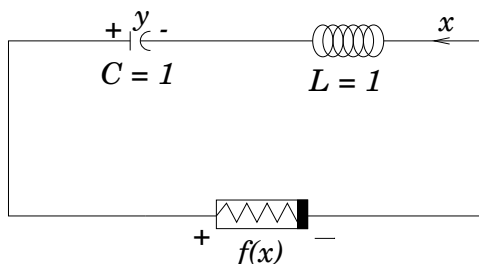


Figure 1: RLC circuit for Problem 2.

function  $v = f(i)$ . Now denote the current through the circuit as  $x$ , and the voltage across the capacitance as  $y$ . The state equations may therefore be written as:

$$\dot{x} = y - f(x) \quad (2)$$

$$\dot{y} = -x \quad (3)$$

We can represent the stored energy in the circuit as a function  $W(x, y) = (x^2 + y^2)/2$ . Show (by studying the rate of change of the stored energy) that if the resistor is strictly passive, meaning that  $x \cdot f(x) \geq 0$  for all  $x$  (with  $x \cdot f(x) = 0$  only when  $x = 0$ ), then the equations admit only one stable equilibrium.

**Problem 6.** Consider the nonlinear second order system

$$\dot{x}_1 = -x_2 + \epsilon x_1(x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \quad (4)$$

$$\dot{x}_2 = x_1 + \epsilon x_2(x_1^2 + x_2^2) \sin(x_1^2 + x_2^2) \quad (5)$$

Show that the linearization is inconclusive to determine stability of the origin. Use the method of Lyapunov and your creative instincts to pick a candidate Lyapunov function  $v(x_1, x_2)$ , and use it to study the stability of the origin between  $\epsilon = 1$  and  $\epsilon = -1$ .

**Problem 7: Proving Stability of a Boundary control scheme.** The equation of motion of a linear

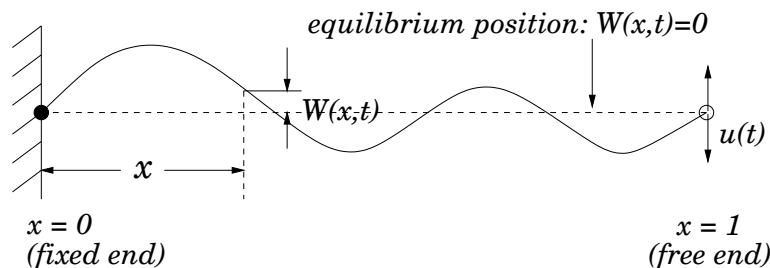


Figure 2: Boundary control stabilization of a vibrating string.

frictionless vibrating string of length 1 is given by the following partial differential equation:

$$m \frac{\partial^2 W(x, t)}{\partial t^2} - T \frac{\partial^2 W(x, t)}{\partial x^2} = 0 \quad (6)$$

where  $W(x, t)$  denotes the displacement of the string at location  $x$  at time  $t$ , with  $x \in (0, 1)$ ,  $t \geq 0$ . The quantities  $m$  and  $T$  are the mass per unit length of the string and the tension in the string, respectively.

Let the string be fixed at  $x = 0$  as shown in Figure 2. Thus,  $W(0, t) = 0$  for all  $t \geq 0$ . Let a vertical force  $u(\cdot)$  be applied to the free end of the string at  $x = 1$ . The balance of the forces in the vertical direction (at  $x = 1$ ) yields:

$$u(t) = T \left. \frac{\partial W(x, t)}{\partial x} \right|_{x=1} \quad (7)$$

for all  $t \geq 0$ . This means that whatever force  $u(\cdot)$  is applied, equation (7) always has to be true.

The energy of the string at time  $t$  is

$$E(t) = \frac{1}{2} \int_0^1 m \left( \frac{\partial W(x, t)}{\partial t} \right)^2 dx + \frac{1}{2} \int_0^1 T \left( \frac{\partial W(x, t)}{\partial x} \right)^2 dx \quad (8)$$

which consists of the kinetic energy (first integral in above) and the strain energy (second integral in the above).

Show that if the following *boundary control*

$$u(t) = -k \left. \frac{\partial W(x, t)}{\partial t} \right|_{x=1} \quad (9)$$

is applied, where  $k > 0$ , then  $\dot{E}(t) \leq 0$  for all  $t \geq 0$ .

(HINT: Integration by parts will come in handy here:  $\int_a^b u dv = uv|_a^b - \int_a^b v du$ .)

(Remarks: The nature of this control law is that of velocity feedback, which generates damping in an inertia system. It can be shown that the string can be stabilized to its equilibrium position by the boundary control. Also,  $E(t) \rightarrow 0$  as  $t \rightarrow \infty$ .)