E209A Analysis and Control of Nonlinear Systems
Problem Set 5

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Problem 1: Switched Control of an Inverted Pendulum. Figure 1 describes the control of an inverted pendulum, where the "switch" can be:

- an ideal relay ($f(e) = 1$ if $e \geq 0$, $f(e) = -1$ if $e < 0$)
- a relay with deadband of width ± 0.1
- a relay with hysteresis of width ± 0.1

For each type of switch, determine equilibria and use phase plane techniques to conjecture the behavior of $\theta(t)$ as $t \to \infty$ (do limit cycles exist?). If you do predict the existence of a limit cycle, can you say anything about its amplitude and frequency?

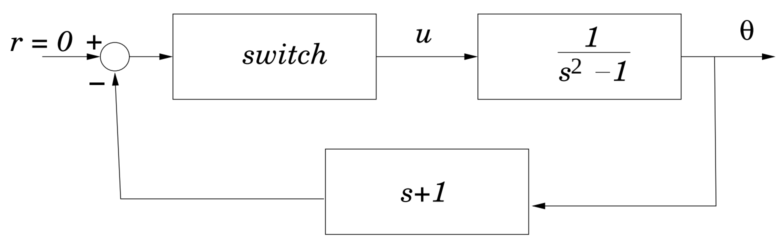


Figure 1: Control loop for Problem 1.

Problem 2: Sliding Mode Control of an Inverted Pendulum. Consider the inverted pendulum of the previous problem.

$$G(s) = \frac{\theta(s)}{u(s)} = \frac{1}{s^2 - 1} \tag{1}$$

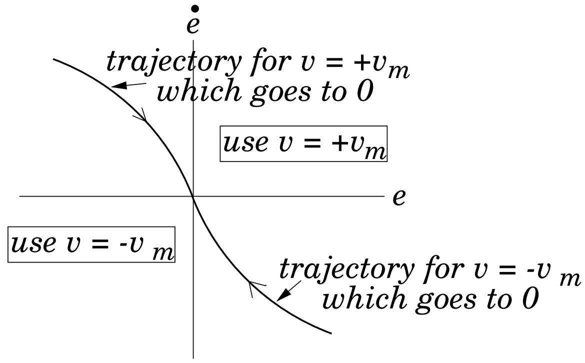
Now, design a sliding mode control law for this system, assuming that your "sliding surface" in the $(\theta, \dot{\theta})$-plane is given by $f = \dot{\theta} + \theta$, and use $\delta = 0.1$ in your sliding control, where δ is the coefficient of the signum function. You can try experimenting with a few other values of δ to see what happens.

Compare your responses to those of the "bang-bang" control with an ideal relay — ie. overplot $\theta(t)$ and $u(t)$ of these two controllers. Also, overplot the phase plane trajectories on the $(\theta, \dot{\theta})$-plane of these two controllers.

Problem 3: Time Optimal Switching Control of DC Motor. Consider the DC motor

$$G(s) = \frac{\theta(s)}{v(s)} = \frac{1}{s(s+1)} \tag{2}$$

from Lecture Notes 7. Derive a mathematical expression for the switched control scheme, in which the switching line corresponds to *actual system trajectories* with $v = +v_m$ and $v = -v_m$ applied, which go to $(0, 0)$ as shown in the following figure.



Simulate the DC-motor with this new switched control law and plot resulting trajectories in the $(e, \dot{e})$-plane. Comment on the advantages and disadvantages of this control law over the sliding mode control that we derived in class.

Problem 4: Describing function determination.

(a) Consider $f(x) = \alpha x$, for all x, for some $\alpha \in \Re$. Determine $N_f(a)$.

(b) Consider a SVSS function f and the function g defined by

$$g(x) = \alpha f(x), \text{ for all } x \tag{3}$$

for some $\alpha \in \Re$. Determine $N_g(a)$ in terms of $N_f(a)$.

(c) Consider the 'threshold' function given in Figure 2. Determine $N_f(a)$ using the graphical method that

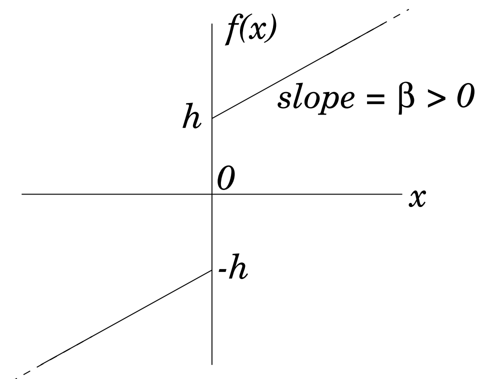


Figure 2: Threshold function for Problem 4.

we discussed in class. Sketch $Re(N_f(a))$ vs. a and sketch the $-\frac{1}{N(a)}$ locus. Now determine $N(a)$ in a more intelligent way, using the known describing functions of part (a) above and of the relay, and considering the properties of parallel connections of nonlinear functions.

Problem 5: Describing function determination and use. An input-output relation in which the output is zero until the magnitude of the input exceeds a certain value, is said to contain a *dead zone*. Such a relation is shown in Figure 3, where the dead zone has width 2δ and the relation has linear gain α outside of this region.

(a) Compute the describing function $N(a)$ for a device with dead zone characteristic shown in Figure 3. ocaption

(b) Now consider this device in the feedback loop configuration of Figure 4, where $G(s)$ is:

$$G(s) = \frac{1}{s(s+1)^2} \tag{4}$$

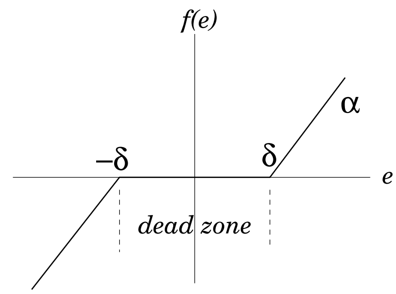


Figure 3: Dead zone characteristic for Problem 5.

having a Nyquist plot as shown in Figure 5, for $p = 0$ where p is the number of poles of $G(s)$ in the right half plane. Does describing function analysis predict sustained oscillation in the error signal $e(t)$? If so,

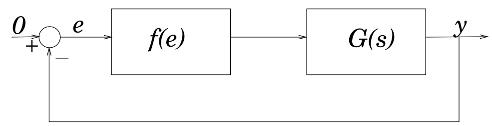


Figure 4: Feedback loop for Problem 5.

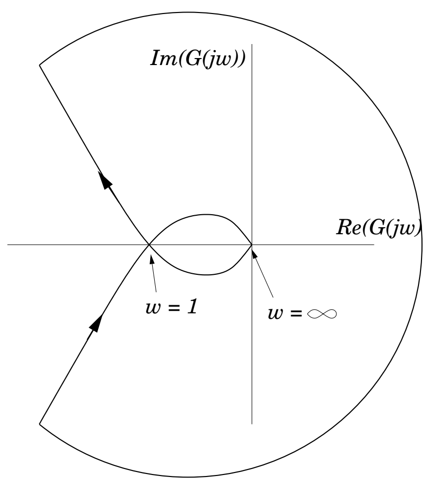


Figure 5: Nyquist plot of $G(s)$ for Problem 5 ($p = 0$).

under what condition is oscillation predicted, and what is the corresponding amplitude and frequency of the predicted oscillation?

Problem 6: Describing function analysis to predict oscillations in Van der Pol. Consider the following equation form of the Van der Pol oscillator:

$$\ddot{y} + \epsilon(3y^2 - 1)\dot{y} + y = 0 \tag{5}$$

In this problem, you will use describing function analysis to attempt to predict whether a limit cycle (oscillation) exists, and with what amplitude and frequency.

(a) Show that this Van der Pol equation may be represented in the feedback form of Figure 6, where

$$f(u) = u^3 \tag{6}$$

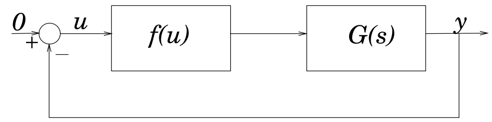


Figure 6: Feedback system for Problem 6.

and

$$G(s) = \frac{\epsilon s}{s^2 - \epsilon s + 1} \tag{7}$$

(b) Use describing function analysis, and the Nyquist plot of $G(j\omega)$, shown in Figure 7(a), to predict whether or not the circuit will oscillate: (i) for $\epsilon \to 0$; (ii) for $\epsilon \to \infty$. In each of the above cases, what is the predicted amplitude and frequency of oscillation?

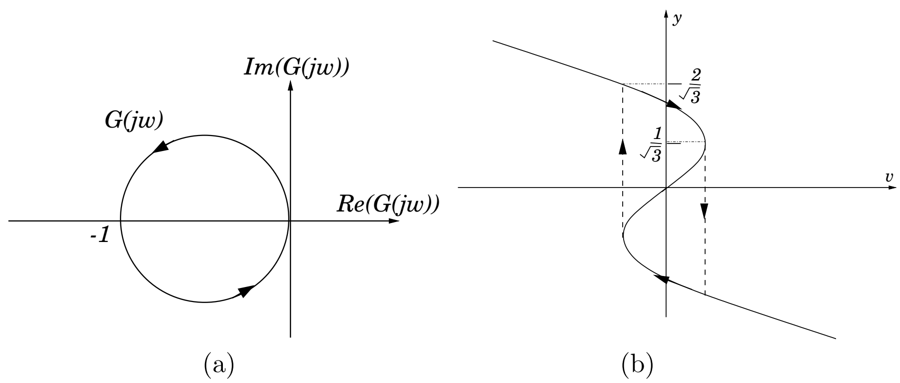


Figure 7: (a) Nyquist plot of $G(j\omega)$, for Problem 6. (b) Showing the curve $v = y - y^3$, for Problem 6.

(c) Now, rewriting (5) as two first order ODEs (where $\tau = t/\epsilon$):

$$\frac{dv}{d\tau} = y \tag{8}$$

$$\frac{dy}{d\tau} = \epsilon^2(-v + (y - y^3)) \tag{9}$$

show that as $\epsilon \to \infty$, $dy/d\tau$ can only remain finite when $v = y - y^3$, meaning that the trajectory of the system consists of segments of this curve (as shown in Figure 7(b)) joined by vertical segments which are traversed instantaneously. Using this fact, the fact that on the curved segments:

$$y = \frac{dv}{d\tau} = (1 - 3y^2)\frac{dy}{d\tau}$$

the y coordinates of the curved trajectory segment are bounded between $1/\sqrt{3}$ and $2/\sqrt{3}$, and that the curve $v = y - y^3$ is an odd function (symmetric through the origin), to show that the period of oscillation (in t) for large ϵ approaches $(3 - 2\ln 2)\epsilon$. How does this compare to the period computed via describing function analysis? Explain any difference, in terms of the frequency response of $G(j\omega)$ for large ϵ.