

E209A Analysis and Control of Nonlinear Systems

Problem Set 4

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Problem 1. Consider the set of differential equations parameterized by (ν_1, ν_2) :

$$\dot{x} = y \tag{1}$$

$$\dot{y} = \nu_1 x + \nu_2 y + x^3 - x^2 y \tag{2}$$

- (a) By solving for the equilibria, determine what kind of bifurcation occurs at $\nu_1 = 0$, for all ν_2 .
- (b) For which of the four quadrants in the (ν_1, ν_2) -plane can you prove that limit cycles do NOT exist? Explain.

Problem 2: Unfolding of a double zero eigenvalue. Conjecture a bifurcation diagram for the two parameter family

$$\dot{x} = y \tag{3}$$

$$\dot{y} = \mu_1 + \mu_2 y + x^2 + xy \tag{4}$$

Please establish bifurcations of equilibria, Hopf bifurcations, and then conjecture other bifurcations to fill in the picture. You can use Matlab for help. The final solution should show representative phase portraits in the (μ_1, μ_2) plane.

Problem 3: A Bifurcation study of a 3-D Pendulum. Consider a bob of mass m which is attached to

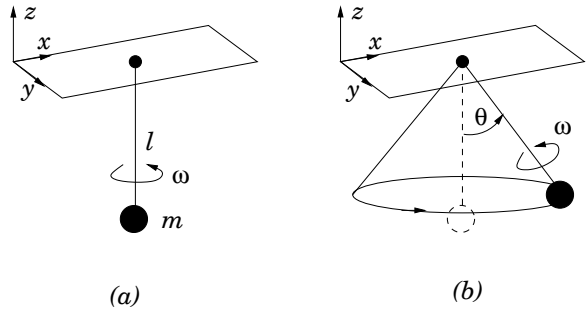


Figure 1: Figure for Problem 3.

a massless rigid rod of length l . The rod is hinged to a moveable joint, and is allowed to hang freely. The rod is then rotated at an angular speed ω about its pivot, so that it rotates in the $-z$ -axis as shown in Figure 1(a). It is observed that as ω increases, a **critical angular speed** $\omega = \omega_0$ is reached at which the rod begins to rise up as it rotates. Thus, the bob and rod system becomes displaced by an angle θ from the vertical, as shown in Figure 1(b).

The nonlinear dynamic equation for θ is:

$$ml\ddot{\theta} = ml\omega^2 \cos \theta \sin \theta - mg \sin \theta \tag{5}$$

Using state variables $x_1 = \theta$, $x_2 = \dot{\theta}$, derive the equilibria of this system. Show that for values of ω less than some critical value, there is only one equilibrium (at least only one that makes physical sense), and for values of ω greater than this critical value, there are additional equilibria. Denote this critical value ω_0 , and derive an expression for ω_0 in terms of g and l . Determine the stability type of the equilibria as ω increases, and then sketch a bifurcation plot in (x_1, x_2, μ) for bifurcation parameter $\mu = \omega - \omega_0$. Finally, explain your results in terms of the observed behavior of the system.

Problem 4: Van der Pol oscillator.

The Van der Pol oscillator is one of the best known models in nonlinear systems theory, originally developed to describe the operation of an electronic valve oscillator, which depends on the existence of a region with effective negative resistance. As we've seen in Problem Set 2, this model is also used to describe the behavior of the pumping heart.

The equation is that of a simple harmonic oscillator with nonlinear damping, written as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(x_2 - x_2^3) \end{aligned}$$

- (a) Determine the stability type of the origin $[x_1, x_2]^T = [0, 0]^T$ for $\epsilon > 0$.
- (b) Fix $\epsilon = 1$. We may conjecture the existence of a limit cycle for the Van der Pol oscillator by simulating the system in Matlab. Here, I would like you to prove that a limit cycle exists. You can do this by using the following steps.

Consider the region shown in Figure 2. The labelled points A, \dots, J are defined as follows, for parameters $a, b \in \mathfrak{R}$, where $b > 0, a > b$:

$$\begin{aligned} A &= (-a, 0) \\ B &= (-a, 1) \\ C &= (-\sqrt{a^2 - b^2}, \sqrt{1 + b^2}) \\ D &= (0, \sqrt{1 + b^2}) \\ E &= (b, 1) \\ F &= (a, 0) \end{aligned}$$

The lower half of the region is symmetric through the origin. The segments AB, CD and EF are straight

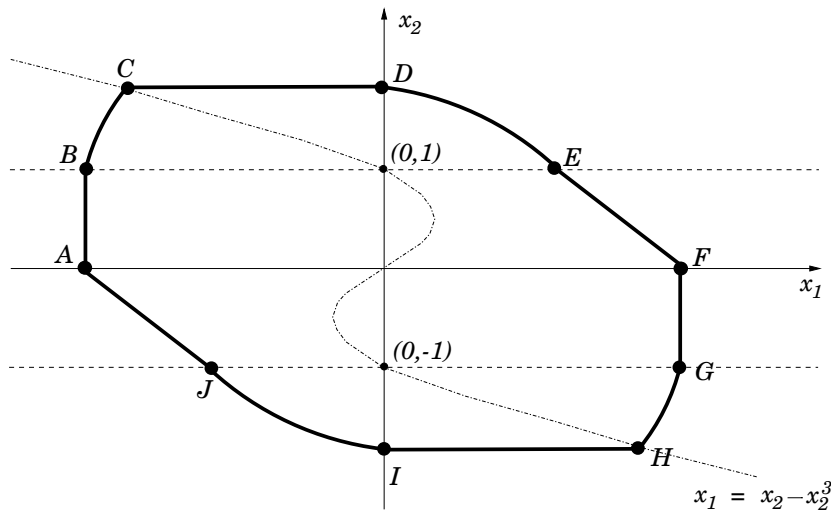


Figure 2: Region for Problem 4.

line segments, BC and DE are circular arcs. The isocline $x_1 = x_2 - x_2^3$ is also shown, as are the two lines $x_2 = 1$ and $x_2 = -1$.

Show that the region enclosed by this boundary is positively invariant for suitable values of a and b (find these values). Then use a modified version of this region to show that a limit cycle exists for the Van der Pol oscillator with $\epsilon = 1$.

Problem 5: This problem is adapted from the Ph.D. Thesis (MIT) of Professor Michael Branicky at Case Western Reserve University.

Consider the two linear systems:

$$\dot{x} = A_1x, \quad \text{and} \quad \dot{x} = A_2x$$

where

$$A_1 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 100 \\ -10 & -1 \end{bmatrix}$$

Note that both linear systems are stable; the individual phase portraits for each system are shown in Figure 3.

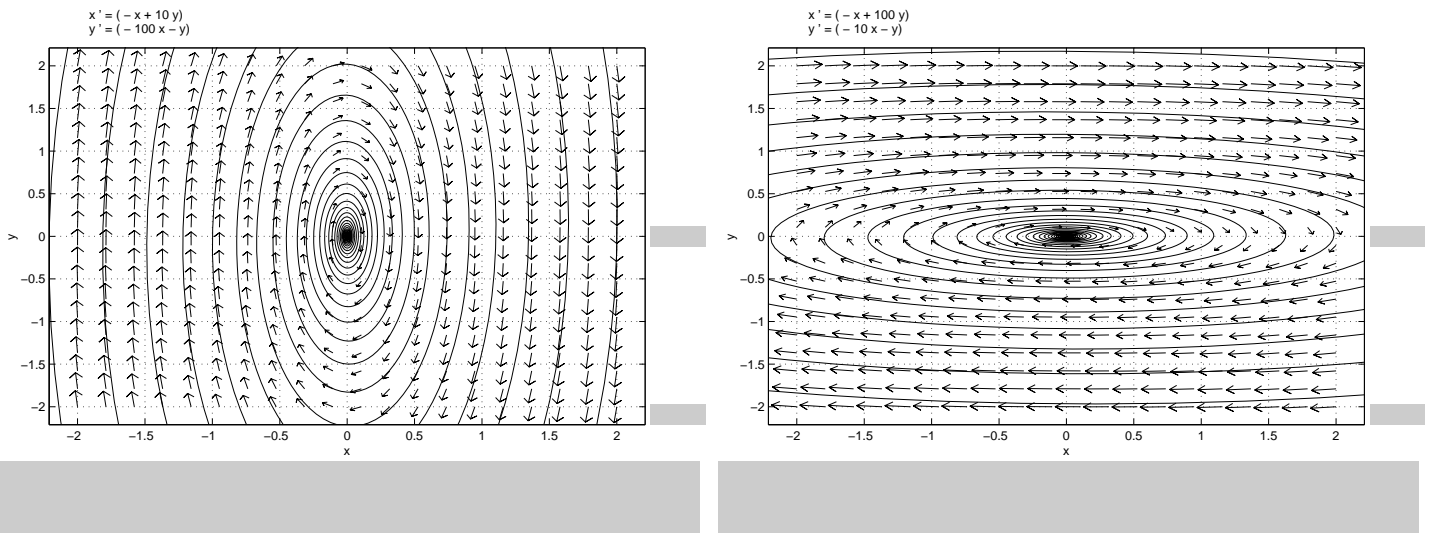


Figure 3: (left) Phase portrait of $\dot{x} = A_1x$; (right) Phase portrait of $\dot{x} = A_2x$. Figure generated using phase plane software from <http://math.rice.edu/polking/odesoft/>.

An interesting piecewise linear system results from switching between these two systems, using switching lines $x_1 = 0$ and $x_2 = 0$, and different switching policies. For example, consider the two switching policies diagrammed in Figure 4: in switching policy (i), $\dot{x} = A_1x$ is valid when $x_1x_2 < 0$ and $\dot{x} = A_2x$ is valid when $x_1x_2 \geq 0$; in switching policy (ii), $\dot{x} = A_1x$ is valid when $x_1x_2 \geq 0$ and $\dot{x} = A_2x$ is valid when $x_1x_2 < 0$.

(a) By sketching the phase portraits resulting from both switching policies above, conjecture which of the two policies is stable. Are both of the policies stable?

(b) Now, by considering how the switched system behaves along level curves of the function

$$V(x) = 2.75x_1^2 + 0.275x_2^2$$

prove your conjecture above.

References

[1] S. S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer-Verlag, 1999.

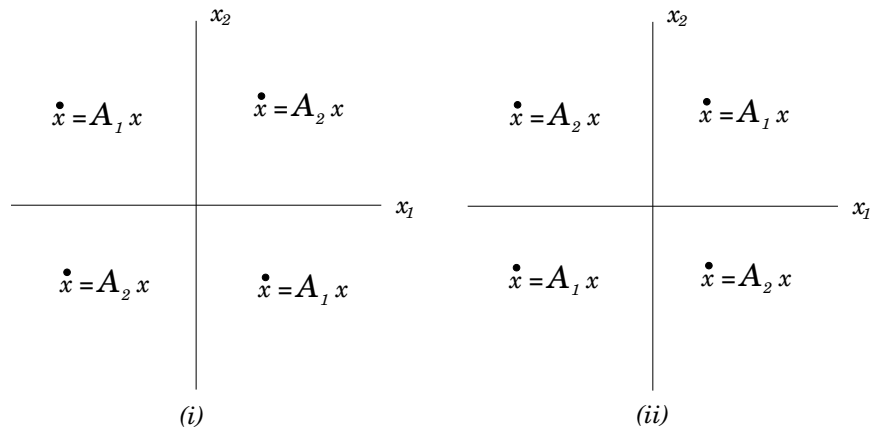


Figure 4: (left) Switching policy (i); (right) Switching policy (ii).