E209A Analysis and Control of Nonlinear Systems
Problem Set 3

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Problem 1: Planar phase portraits. Shown in Figure 1 are three phase portraits of planar systems. Which of them are correct, and which are incorrect? Why? Modify the incorrect ones, not by deleting any orbits, but by changing their stability type or adding new orbits.

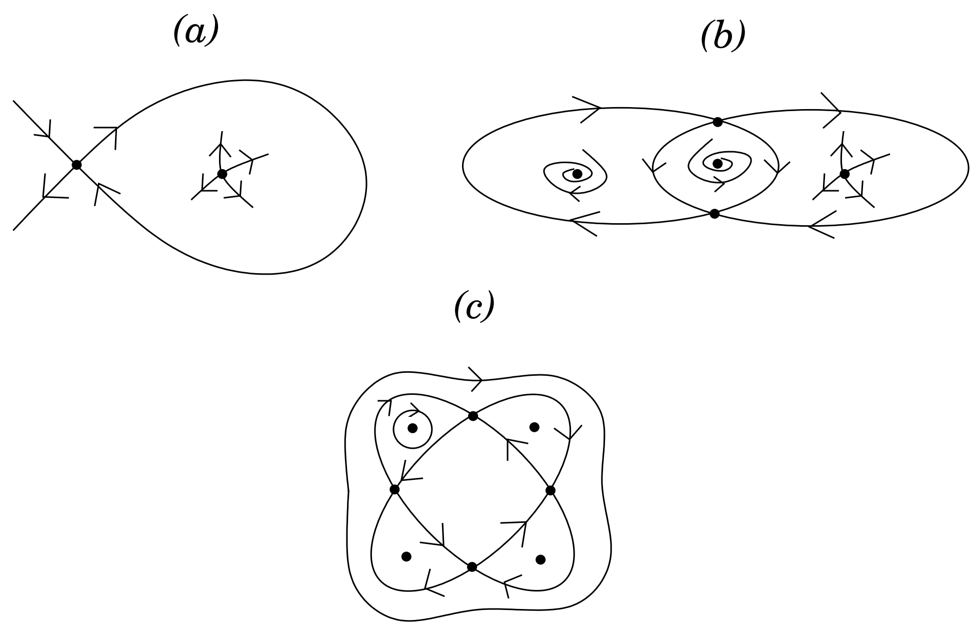

Figure 1: Phase portraits for Problem 1.

Problem 2. Consider the system

$$\dot{x}_1 = x_2 \quad (1)$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - 2x_2^2) \quad (2)$$

Prove that this system has a closed orbit.

Problem 3. Prove that each of the following systems has no limit cycles:

$$(i) \quad \begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= g(x_1) + ax_2, \text{ where } a \text{ is a constant}, a \neq 1 \end{aligned}$$

$$(ii) \quad \begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_2 \end{aligned}$$

Problem 4. One popular model for the propagation of disease through a population is given by the following,

$$\ddot{x} + x(1 - x) + 2\dot{x} = 0 \quad (3)$$

Here, x is the density of the population with the disease, where $0 \leq x \leq 1$. Denote $x_1 = x$, $x_2 = \dot{x}$.

(a) Determine the equilibria of this model and their stability type.

(b) Prove that the closed region enclosed by the lines $x_2 = 0$, $x_1 = 1$, and $x_2 = -x_1$ is positively invariant.

(c) Use parts (a) and (b) to prove that this system has a solution $x(t)$ such that $x(0) = 1$, $x(\infty) = 0$, and $\dot{x}(t) < 0$ for all $t \in [0, \infty)$.

Problem 5: Lotka-Volterra Predator-Prey Equations [2]. Vito Volterra was an Italian mathematician (1860-1940), who developed a mathematical model to explain the results of a statistical study of fish populations in the Adriatic Sea. In particular, his model explains the increase in predator fish (and corresponding decrease in prey fish) which he observed during the World War I period. Volterra produced a series of models for the interaction of two or more species. Alfred J. Lotka was an American biologist and actuary who independently produced many of the same models.

One of the simplest of their models takes the form

$$\dot{x} = ax - bxy \tag{4}$$

$$\dot{y} = -dy + cxy \tag{5}$$

where $x > 0$ denotes the sardine (prey) population and $y > 0$ denotes the shark (predator) population. a, b, c, and d are all positive constants. Note that the equations model the facts that: sardines multiply faster as they increase in number; the number of sardines decreases as both the sardine and shark population increases; sharks increase in number at a rate proportional to the number of shark-sardine encounters. Analyze the equilibria of this system (their stability and type), and show (using simulation, and analysis if possible) that for different values of a, b, c, and d, the model predicts both cyclic variations in population as well as convergence to steady state values.

Now repeat this analysis for a more complicated model, in which the sardine population saturates in the absence of sharks, and vice versa.

$$\dot{x} = (a - by - \lambda x)x \tag{6}$$

$$\dot{y} = (-d + cx - \mu y)y \tag{7}$$

where $\lambda, \mu > 0$.

Problem 6. A model that is used to analyze a class of experimental systems known as chemical oscillators is given by:

$$\begin{aligned} \dot{x}_1 &= a - x_1 - \frac{4x_1x_2}{1+x_1^2} \\ \dot{x}_2 &= bx_1\left(1 - \frac{x_2}{1+x_1^2}\right) \end{aligned} \tag{8}$$

where x_1 and x_2 are dimensionless concentrations of certain chemicals and a, b, are positive constants.

(a) Find the equilibrium (equilibria) of this system.

(b) Show that the region

$$M = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0, x_1 \leq a, x_2 \leq 1 + a^2\}$$

is invariant. Use the Poincaré-Bendixson Theorem to show that the system has a periodic orbit when $b < 3a/5 - 25/a$.

Problem 7: Analysis of the Lorenz attractor.

The Lorenz attractor is one of the most widely studied examples of chaotic behavior in ODEs: it was originally studied in connection with turbulent convection in fluids, and has also been used to explain irregular spiking in lasers. The dynamics may be modeled as:

$$\dot{x}_1 = \sigma(x_2 - x_1) \tag{9}$$

$$\dot{x}_2 = (1 + \lambda - x_3)x_1 - x_2 \tag{10}$$
$$\dot{x}_3 = x_1 x_2 - b x_3 \tag{11}$$

where σ, λ, and b are positive constants. (In the fluid problem, x_1 represented a Fourier component of the fluid velocity field, and x_2 and x_3 were Fourier components of the temperature gradient.)

(a) Show that this system has equilibria at $(0, 0, 0)$, $(\sqrt{b\lambda}, \sqrt{b\lambda}, \lambda)$, and $(-\sqrt{b\lambda}, -\sqrt{b\lambda}, \lambda)$. Show that the equilibrium point at the origin is unstable, and that the other two equilibria are unstable if:

$$\sigma > b + 1 \tag{12}$$
$$\lambda > \frac{(\sigma + 1)(\sigma + b + 1)}{\sigma - b - 1} \tag{13}$$

(b) Simulate the system for parameters $\sigma = 10, \lambda = 24, b = 2$, and describe the qualitative behavior of trajectories starting close to one of the equilibria away from the origin.

(c) Now, by considering the function:

$$V(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2 + (x_3 - a)^2) \tag{14}$$

(where $a = \sigma + \lambda + 1$) and the rate of change of V along trajectories of the system, show that all trajectories are ultimately contained within a certain bounded region (can you find an expression for this region in terms of the function V?)

References

[1] P. Fife. *Mathematical aspects of reaction-diffusion systems.* Springer-Verlag, Lecture notes in Mathematics, Vol. 28, 1979.

[2] S. S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control.* Springer-Verlag, 1999.