E209A Analysis and Control of Nonlinear Systems
Problem Set 2

Professor C. Tomlin
Department of Aeronautics and Astronautics, Stanford University
Winter 2007
Issued 1/18; Due 1/25

Problem 1: The method of isoclines [1]. A simple but useful technique for the approximation of solution curves in the phase plane is provided by the method of isoclines. Given the nonlinear system

$$\frac{dx_1}{dt} = f_1(x_1, x_2) \tag{1}$$
$$\frac{dx_2}{dt} = f_2(x_1, x_2) \tag{2}$$

we can write (assume for the moment that $f_1(x_1, x_2) \neq 0$)

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \tag{3}$$

We seek curves $x_2 = h(x_1)$ on which the slope $dx_2/dx_1 = c$ is constant. Such curves, called *isoclines*, are given by solving the equation

$$f_2(x_1, x_2) = c f_1(x_1, x_2) \tag{4}$$

Now consider the following system:

$$\dot{x}_1 = x_1^2 - x_1 x_2 \tag{5}$$
$$\dot{x}_2 = -x_2 + x_1^2 \tag{6}$$

- Find the equilibria of this system.
- Show that the x_2-axis is invariant and that the slope dx_2/dx_1 is infinite (vertical) on this line; find other lines in the plane on which the slope dx_2/dx_1 is infinite.
- Now seek isoclines on which $dx_2/dx_1 = c$ for finite c (try $c = 0, .5, 1, 2$)
- Sketch these curves, and the associated slopes dx_2/dx_1 on top of these curves, in the (x_1, x_2) plane.
- Conjecture the phase portrait from this information. (You may want to plug the system into MATLAB to see if your conjecture is correct.)

Problem 2: The Pumping Heart [2, 3]. The human heart, a pump which takes re-oxygenated blood from the lungs and sends it out to the rest of the body, may be modeled (in a very simple form) as an oscillator. The system oscillates between two states: *diastole*, or relaxed state, and *systole*, or contracted state. An electro-chemical stimulus causes the heart muscle to contract and transition from diastole to systole states. A simplified model of this process is the Van der Pol oscillator:

$$\dot{x} = v - \mu(x^3/3 - x) \tag{7}$$
$$\dot{v} = -x \tag{8}$$

where x is the muscle fiber length in the heart, v is the stimulus, and $\mu > 0$ is a parameter. Determine the equilibrium and its stability; plot phase plane portraits for both small and large μ. With reference to these plots, show that in the transition from diastole (long fibers) to systole (short fibers), the contraction happens slowly at first (this ensures no backflow which could damage the heart) but that at a high enough stimulus the fibers contract suddenly to push the blood all throughout the body.

Problem 3: Modification of Duffing's equation [3]. Consider the modified Duffing equation

$$\dot{x}_1 = x_2 \tag{9}$$
$$\dot{x}_2 = x_1 - x_1^3 - \delta x_2 + x_1^2 x_2 \tag{10}$$

Find its equilibria. Linearize about the equilibria. Apply Bendixson's theorem to rule out regions of limit cycles. Synthesize this information to conjecture plausible phase portraits of the system.

Problem 4. First Integrals. One way of studying differential equations in the plane

$$\dot{x} = f(x) \qquad x \in \Re^2 \tag{11}$$

is to attempt to find scalar functions $V : \Re^2 \to \Re$ such that

$$\frac{d}{dt}V(x) = \frac{\partial V}{\partial x}(x) \cdot f(x) = 0 \tag{12}$$

meaning that $V(x)$ is constant along trajectories of (11). Such a function $V(x)$ is called a *first integral* of the motion of the system.

Now consider the nonlinear differential equation

$$\ddot{\theta} = 1 - 2\sin\theta \tag{13}$$

Determine the equilibria of this system and their stability type. In particular, show that some of the equilibria correspond to nonlinear centers, by finding a first integral for this system. SKETCH an approximate phase portrait for (13).

Problem 5: Reaction-diffusion equations [4]. Consider the following *reaction-diffusion system*, in which x_1 is the concentration of chemical A and x_2 is the concentration of chemical B. Chemical A reacts positively (concentration increases) to higher concentrations of chemical B (reaction) and negatively to higher concentrations of itself (diffusion). Likewise, chemical B reacts positively to higher concentrations of chemical A and negatively to higher concentrations of itself.

$$\dot{x}_1 = 2(x_2 - x_1) + x_1(1 - x_1^2) \tag{14}$$
$$\dot{x}_2 = -2(x_2 - x_1) + x_2(1 - x_2^2) \tag{15}$$

Find the equilibria and determine their stability. Does the system have limit cycles?

Problem 6: Hamiltonian Systems.

A *Hamiltonian system* is one in which

$$\dot{x}_1 = \frac{\partial H(x_1, x_2)}{\partial x_2} \tag{16}$$
$$\dot{x}_2 = -\frac{\partial H(x_1, x_2)}{\partial x_1} \tag{17}$$

for some *Hamiltonian function* $H(x_1, x_2)$.

Consider the Duffing equation with $\delta = 0$:

$$\ddot{x} - x + x^3 = 0 \tag{18}$$

Show that for this system (with $x_1 = x$, $x_2 = \dot{x}$), the linearization around the equilibria $(-1, 0)$ and $(1, 0)$ cannot predict the behavior of the nonlinear system around these equilibria. Even so, if we simulate this nonlinear system, we observe what look like closed orbits around these equilibria. Prove that $(-1, 0)$ and $(1, 0)$ *are* in fact centers for the nonlinear system (18), by recognizing that (18) is Hamiltonian, and determining the Hamiltonian function $H(x_1, x_2)$. (HINT: Determine $\dot{H}(x_1, x_2)$ along trajectories of the system.)

References

[1] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983.

[2] E. Beltrami. *Mathematics for Dynamic Modeling, 2nd edition*. Prentice Hall, 1998.

[3] S. S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control*. Springer-Verlag, 1999.

[4] P. Fife. *Mathematical aspects of reaction-diffusion systems*. Springer-Verlag, Lecture notes in Mathematics, Vol. 28, 1979.