E209A Analysis and Control of Nonlinear Systems
Problem Set 1

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Problem 1: Bowing of a violin string with different models of stiction [1, 2]. Consider the Rayleigh model of a violin string. The mechanical model consists of a mass M and a spring with spring constant k, on a conveyor belt moving with constant velocity b. $F_b(\cdot)$ models sticky friction, or "stiction", as shown in Figure 1.

$$M\ddot{x} + F_b(\dot{x}) + kx = 0 \tag{1}$$

Model each arm of "stiction 1" as a quadratic: the upper arm is modeled as $((y-b)-c)^2 + d$ and the lower arm as $-((y-b)+c)^2 - d$ with parameters $b = 1$, $c = 2$, $d = 3$ (and $y = \dot{x}$). Assume that $M = 3$, $k = 3$. Calculate the equilibrium in $(x, \dot{x})$ and determine its stability. Now show what happens to this equilibrium (and its stability) as the speed of the belt is increased: try $b = 2$, $b = 2.1$. Use MATLAB to plot $\dot{x}$ vs. x, for a few initial conditions $(x(0), \dot{x}(0))$, for each value of b.

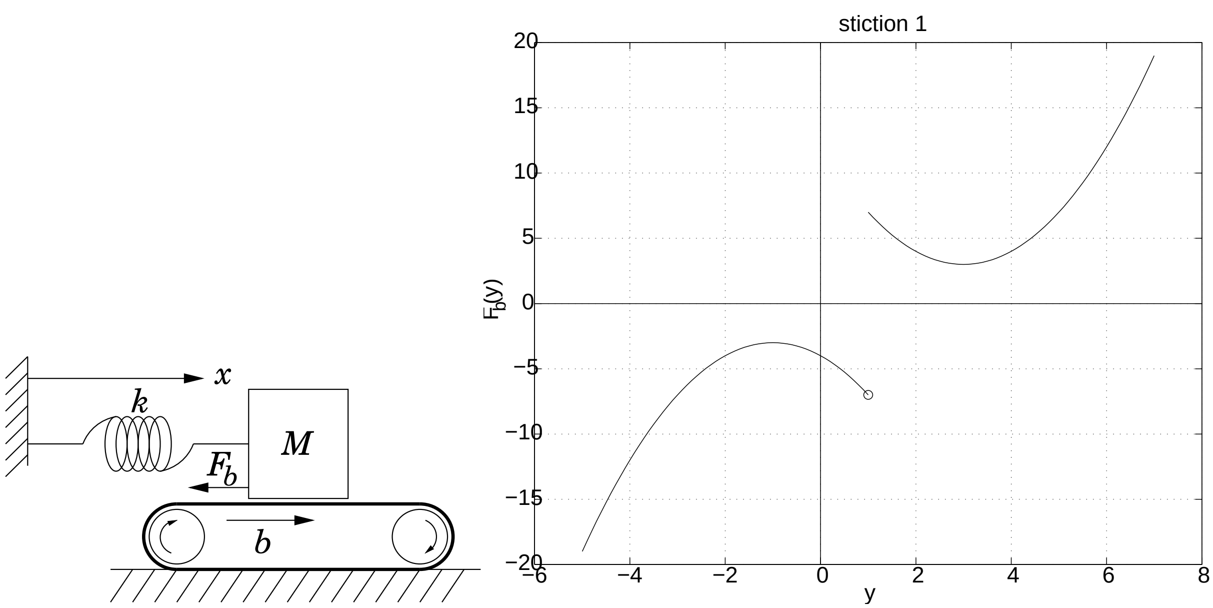

Figure 1: Rayleigh model of a violin string with stiction characteristic as given.

Now suppose the conveyor belt were replaced with one having a stiction characteristic shown in Figure 2: "stiction 2". Calculate the equilibrium in $(x, \dot{x})$ and determine its stability for $b = 1$, $M = 1$, $k = 1$. Use MATLAB to plot $\dot{x}$ vs. x, for a few initial conditions $(x(0), \dot{x}(0))$, and explain intuitively what is happening in your plots.

Problem 2: Models for surge in jet engine compressors [3]. A simple second order model for explaining surge in jet engine compressors (which can cause loss of efficiency and sometimes stall) is given by

$$\dot{x} = B(C(x) - y) \tag{2}$$

$$\dot{y} = \frac{1}{B}(x - F_\alpha^{-1}(y)) \tag{3}$$

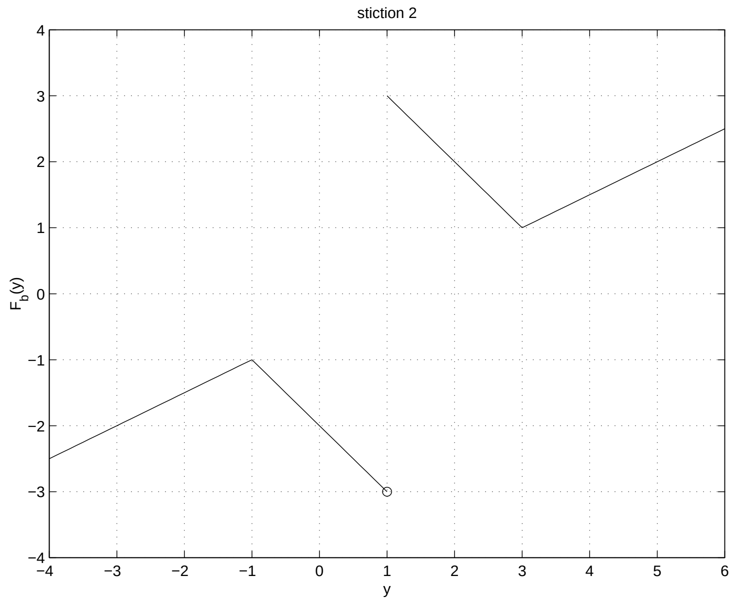

Figure 2: A new stiction characteristic.

Here x stands for the non-dimensional compressor mass flow, y represents the plenum pressure rise in the compressor unit, B stands for the non-dimensional compressor speed, and α, the throttle area. For definiteness, we will assume the functional form of the compressor characteristic

$$C(x) = -x^3 + (3/2)(b+a)x^2 - 3abx + (2c + 3ab^2 - b^3)/2 \tag{4}$$

and the functional form of the throttle characteristic

$$F_\alpha(x) = (x^2/\alpha^2)\mathrm{sign}(x) \tag{5}$$

where

$$\mathrm{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{6}$$

Here, $a,\ b,\ c$ are positive constants determining the qualitative shape of the compressor characteristic. Linearize the system about its equilibria and characterize the stability of the equilibria. In particular show that an equilibrium at (x^*, y^*) is asymptotically stable provided that $x^* \notin (a, b)$. For $x^* \in (a, b)$ determine the stability as a function of the parameters B, α.

Show, using MATLAB simulations in the (x, y) plane for various initial conditions $(x(0), y(0))$, the effect of increasing the compressor speed B. (Some good parameters to use are $a = 1$, $b = 3$, $c = 6$, $\alpha = 1$, and $B = .1, B = .3, B = 1$).

BONUS: Explain, with reference to the equations (2,3) the effect that you are noticing for large B.

Problem 3: Degenerate phase portraits. Consider the second-order linear system

$$\dot{x} = Ax, \text{ for } x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Re^2 \tag{7}$$

and $A \in \Re^{2\times 2}$. (Note that x_1 and x_2 are general variables in the system, ie. x_2 is not necessarily equal to $\dot{x}_1$). Let λ_1, λ_2 be the eigenvalues of A. Analyze the situations (i) $\lambda_1 = 0$, $\lambda_2 < 0$; (ii) $\lambda_1 = 0$, $\lambda_2 = 0$, by determining the equilibria in each case, sketching phase plots, and explaining your results.

Problem 4: SR Latch. A dynamical model of a standard SR latch may be given as

$$\dot{x}_1 = \mathrm{NOT}(x_2) - x_1 \tag{8}$$
$$\dot{x}_2 = \mathrm{NOT}(x_1) - x_2 \tag{9}$$

where the function NOT is drawn below (the state variables x_1 and x_2 correspond to capacitor voltages in the latch). Sketch an approximate phase portrait showing the three equilibria of the system.

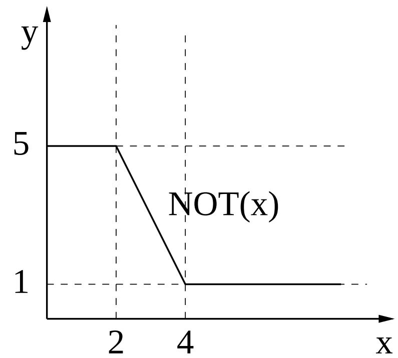

Figure 3: NOT

References

[1] J.W.S. Rayleigh. *The Theory of Sound, Volume 1.* Dover, 1945. Originally published in 1877.

[2] S. S. Sastry. *Nonlinear Systems: Analysis, Stability, and Control.* Springer-Verlag, 1999. Chapter 1.

[3] S. M. Oliva and C. N. Nett. A general nonlinear analysis of a second-order one-dimensional, theoretical compression system model. In *Proceedings of the American Control Conference, Atlanta, GA*, pages 3158–3165, June 1991.