

E209A Analysis and Control of Nonlinear Systems

Midterm Practice Problems

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Here are some problems from old midterms that you can use as practice: Problems 1-3 came from a single 1.5 hour midterm, Problem 4 was part of a take-home midterm (so MATLAB/more time was available).

In the midterm, you must quote the theorems and results from class that you are using in your solutions, however there is no need to reprove them.

Problem 1: (10 points) The following system is used to model oscillations in biochemical interactions:

$$\begin{aligned}\dot{x}_1 &= k_1 x_2 \left(1 - \frac{x_1}{1+x_2}\right) \\ \dot{x}_2 &= k_2 - x_2 - \frac{4x_1 x_2}{1+x_2}\end{aligned}\tag{1}$$

where x_1 and x_2 represent the concentrations of two biochemical components; and $k_1 > 0$, $k_2 > 0$ are positive constants.

- (a) Find the equilibria of this system.
- (b) Derive a condition under which the system is guaranteed to have a periodic orbit inside the region

$$K = \{(x_1, x_2) | x_1 \geq 0, x_2 \geq 0, x_1 \leq 1 + k_2^2, x_2 \leq k_2\}$$

- (c) Using this, describe the type of bifurcation that occurs as a function of the parameter k_1 .

Problem 2: (10 points)

Consider the following two systems. For each, provide a proof showing whether or not the system has a closed orbit.

$$\begin{aligned}(a) \quad \dot{x}_1 &= 1 - x_1 x_2^2 \\ \dot{x}_2 &= x_1 \\ (b) \quad \dot{x}_1 &= x_2 \cos x_1 \\ \dot{x}_2 &= \sin x_1\end{aligned}$$

Problem 3: (10 points)

- (a) Consider the system

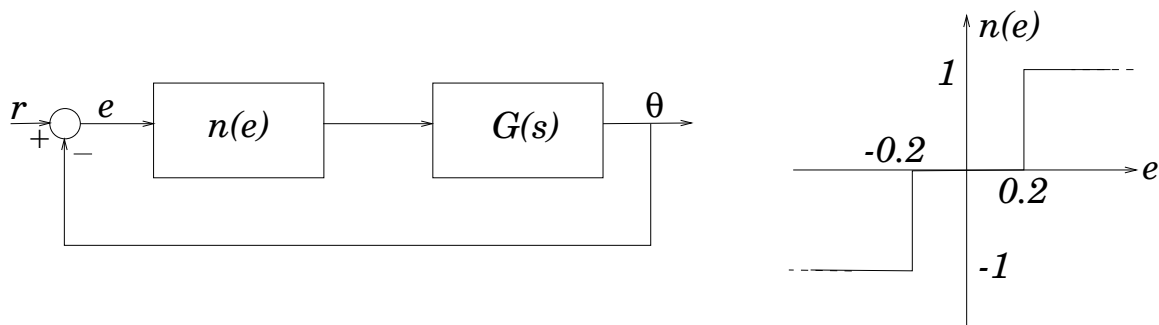
$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u\end{aligned}\tag{2}$$

where the control input u can take the values ± 1 .

- (i) Sketch the phase portrait when $u = 1$.
- (ii) Sketch the phase portrait when $u = -1$.

(iii) By superimposing the two phase portraits, develop a strategy for switching the control between ± 1 so that a trajectory starting from any point in the state space converges to the origin.

(b) Now consider the control system in the Figure below, where $n(e)$ is defined as illustrated. Suppose that r is a unit step function.



Suppose that $G(s) = \frac{1}{s^2}$.

Consider the region of the (e, \dot{e}) phase space corresponding to $-2 \leq e \leq 2$ and $-2 \leq \dot{e} \leq 2$. Determine the isoclines corresponding to $\frac{d\dot{e}}{de} = \pm \frac{1}{2}, \pm 1, \pm \infty$ when $|e| > 0.2$. Hence sketch the trajectory (in that phase-space) corresponding to the initial conditions $e(0) = 0.2, \dot{e}(0) = 2$. It corresponds to a sustained oscillation. Determine the period of oscillation.

Problem 4: (20 points).

Due to your sky-rocketing heating bills, you put on a sweater and turn the thermostat in your house down to $50^\circ F$. The heating system in your house is controlled by a relay with hysteresis, and may be described by the following 1st-order differential equation:

$$\dot{T} + T = f(T)$$

where T is the temperature in your house, $f(T)$ is a double valued relation describing the hysteresis element, and is shown in Figure 1(a). Let $\Delta = 10$ degrees, and let $P_{max} = 100$ be the maximum heater power.

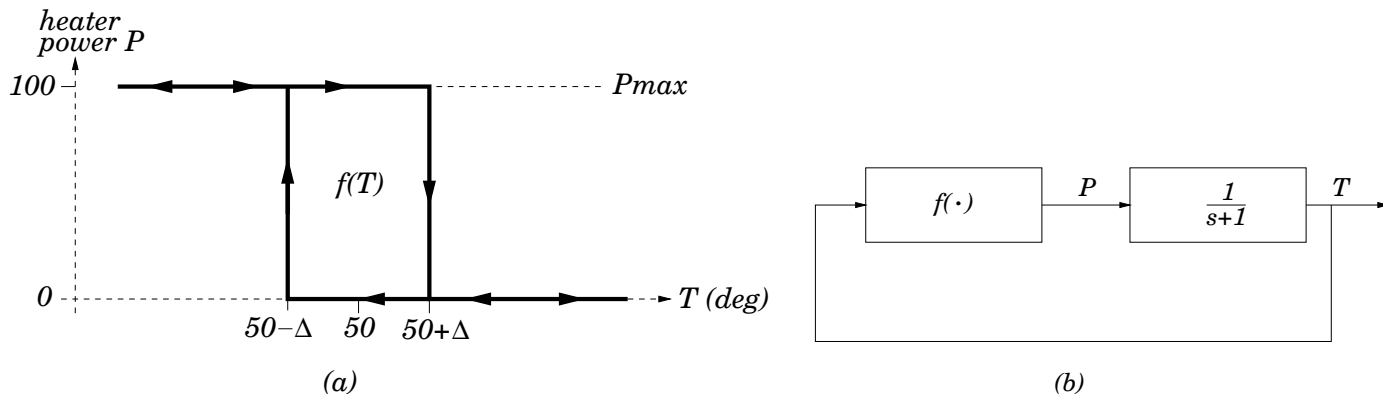


Figure 1: (a) Hysteresis element $f(T)$, assume $\Delta = 10$ degrees. (b) Heating system incorporating $f(T)$.

(a) By direct integration of the system equations for both ‘heater on’ and ‘heater off’, determine whether or not the temperature $T(t)$ of your house will oscillate.

(b) Now, use describing function analysis to predict whether or not the temperature $T(t)$ will oscillate. In order to use the describing function methodology that we studied in class, you can use the following steps:

(i) Show that the system described above (pictured in Figure 1(b)) is equivalent to the system pictured in Figure 2(b), where the error $e = 50 - T$, and the nonlinear element $n(-e)$ is given in Figure 2(a).

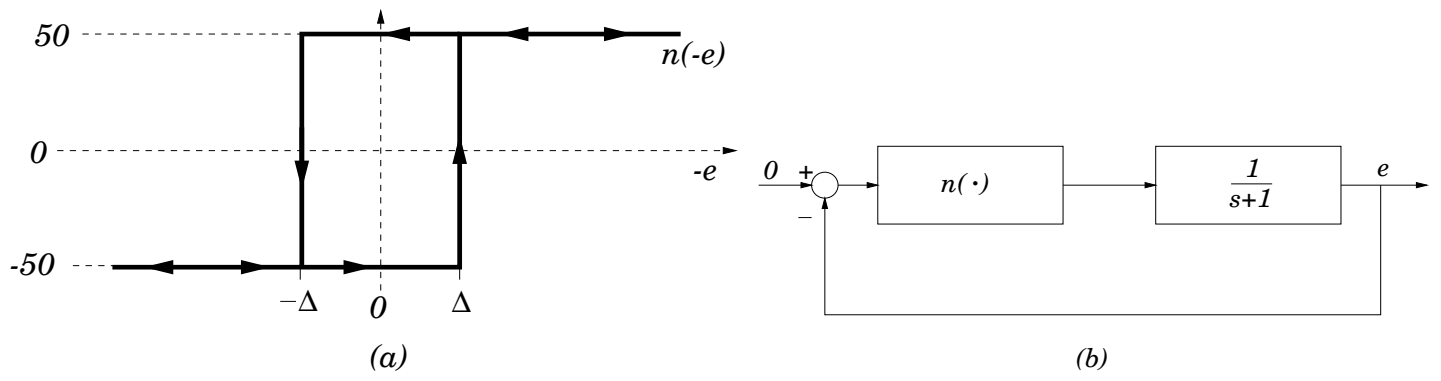


Figure 2: (a) Transformed hysteresis element, $\Delta = 10$ degrees. (b) Transformed heating system.

(ii) Use the methodology studied in class to predict whether or not the error $e(t)$ will oscillate. How good is the prediction from describing function analysis in this case?