

E209A Analysis and Control of Nonlinear Systems

Practice Problems for the Final Exam

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Winter 2007

Here are some problems culled from old finals.

Please quote the theorems and results from class and from the text that you are using in your solutions (no need to reprove them).

Problem 1: Nonlinear Control of the Ball and Beam System.

Consider the ball and beam system shown in Figure 1 below.

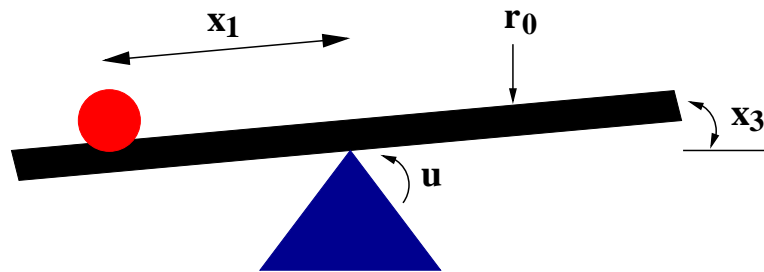


Figure 1: A diagram of the ball and beam apparatus.

The dynamics of the system are given by the following:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= B(x_1 x_4^2 - G \sin x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \\ y &= x_1 - r_0\end{aligned}\tag{1}$$

where x_1 is the position of the ball on the beam, x_2 is the linear velocity of the ball along the beam, x_3 is the angle in radians that the beam makes with the horizontal, and x_4 is the angular velocity of the beam. The input u represents a motor torque input to the beam as shown in the figure, and the output of this system is the ball position relative to a fixed position along the beam, r_0 .

To simplify the problem, we make the assumption that the ball always maintains contact with the beam (even if the beam flips over).

Our goal is to have the ball position $x_1(t)$ track a desired position trajectory $y_D(t)$.

What is the relative degree of this system? Compute the feedback linearizing control law from input u to output y and derive the zero dynamics of this system. What problems do you foresee with direct use of this control law, and discuss how you would remedy them.

Problem 2: Domains of Attraction.

Consider the damped nonlinear oscillator

$$\ddot{y} + 2\zeta\dot{y} + (1 - y)y = 0 \quad (2)$$

where ζ is a constant, with $0 < \zeta < 1$.

(a) Using the state variable definition, $x_1 = y$, $x_2 = \frac{\dot{y} + \zeta y}{\gamma}$, where $\gamma = \sqrt{1 - \zeta^2}$, find an estimate of the domain of attraction of the equilibrium at the origin $(x_1, x_2) = (0, 0)$, using the indirect method of Lyapunov. Where is the other equilibrium point and what is its stability type?

(b) Now, obtain an estimate of the domain of attraction of the origin, using the Lyapunov function $V = y^2 - \frac{2}{3}y^3 + \dot{y}^2$. Compare this with the domain that you computed in part (a).

Problem 3: Model Matching.

Consider the SISO nonlinear system:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \quad (3)$$

and the SISO linear reference model

$$\begin{aligned} \dot{x}_m &= A_m x_m + b_m r \\ y_m &= c_m x_m \end{aligned} \quad (4)$$

Here $x \in \mathbb{R}^n$, $x_m \in \mathbb{R}^{n_m}$. You are told that the nonlinear system has strict relative degree γ at $x = 0$. Find the control law u depending on x , x_m , and r , required to get $y(t)$ to track $y_m(t)$. In particular, what conditions on the reference model are needed to prevent the appearance of derivatives of r in this control law?

Problem 4. Consider the equation:

$$\ddot{y} + h(y)\dot{y} + g(y) = 0 \quad (5)$$

where g and h are differentiable, with continuous derivatives.

(i) Find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(ii) Now, find further conditions on g and h to ensure that the origin is asymptotically stable.

Problem 5: Krasovskii's Method.

Consider the system $\dot{x} = f(x)$ with $f(0) = 0$. Assume that $f(x)$ is continuously differentiable and its Jacobian $[\partial f / \partial x]$ satisfies:

$$P \left[\frac{\partial f}{\partial x}(x) \right] + \left[\frac{\partial f}{\partial x}(x) \right]^T P \leq -I, \quad \forall x \in \mathbb{R}^n, \text{ where } P = P^T > 0 \quad (6)$$

(a) Using the representation $f(x) = \int_0^1 \frac{\partial f}{\partial x}(\sigma x) x d \sigma$, show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x, \quad \forall x \in \mathbb{R}^n \quad (7)$$

(b) Show that the origin is globally asymptotically stable.

Problem 6: Regions of Attraction.

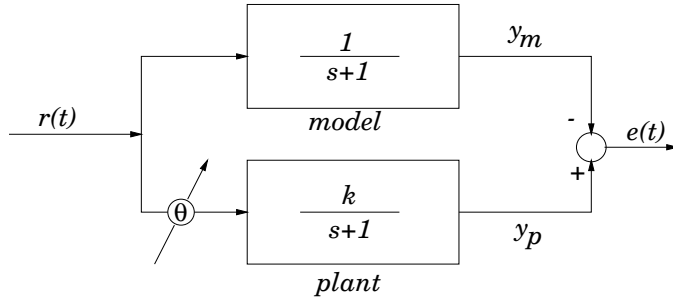
Consider the system:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3 + x_2 \\ \dot{x}_2 &= 3x_1 - x_2 \end{aligned} \quad (8)$$

(a) Find all equilibria and determine their stability type.

(b) Estimate the region of attraction of one of the asymptotically stable equilibrium points. HINT: it may be easier to do this by defining a translated set of coordinates (and dynamics) with the equilibrium point at the origin in this new frame.

(c) Construct the phase portrait of the system (using MATLAB) and indicate on each the exact region of attraction, as well as your estimates.



Problem 7: Adaptive Control of an Unknown Plant.

You are given a single-input single-output plant of the form $\frac{k}{1+s}$ (k is unknown). You would like to add a feedforward control θ so that the transfer function resembles the model. If you knew k , you would set $\theta^* = 1/k$. Since you do not know k , you start with an estimate $\theta(0)$ and update it depending on the error between $y_p(t)$ and the desired model output $y_m(t)$. Define $\phi(t) := \theta(t) - \theta^*$ to be the parameter error.

(a) Write the differential equations for $y_p(t)$ and $y_m(t)$ and subtract them to get the differential equation relating e to ϕ .

(b) Your job is to choose the update law $\dot{\phi}(t) = \dot{\theta}(t)$. Use the differential equation for e (from above) to make $V = \frac{1}{2}e^2 + \frac{1}{2}k\phi^2$ a Lyapunov function for the (e, ϕ) system. What do you need to assume about k to make the candidate function positive definite? Note that $\dot{\phi}(t)$ cannot depend on k or ϕ or θ^* (since they are unknown), but can depend on θ , y_p , y_m , e , and r . What is the most you can say about the stability of $\dot{\phi}(t)$?