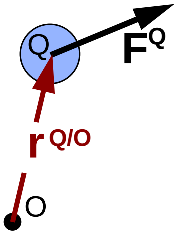
Chapter 14

Moments and torque

14.1 Summary of moments and torques

Expression	Description
$\mathbf{M}^{\mathbf{F}^Q/O} \triangleq \mathbf{r}^{Q/O} \times \mathbf{F}^Q$	Moment of force $\mathbf{F}^Q$ about point O
$\mathbf{M}^{S/O} \triangleq \sum_{i=1}^{n} \mathbf{M}^{\mathbf{F}^{Q_i}/O}$	Moment of a set S of forces about point O
$\mathbf{M}^{S/P} = \mathbf{M}^{S/O} + \mathbf{r}^{O/P} \times \mathbf{F}^S$	Shift theorem for moment of a force
$\mathbf{T}^S \triangleq \mathbf{M}^{S/O}$	Torque is the moment of a set S of forces whose resultant is $\mathbf{0}$.

14.2 Moment of a vector



The ***moment of a vector*** results from the cross product of a position vector with a ***bound vector***.[a] The moment of the vector $\mathbf{F}^Q$ (bound to point Q) about point O is denoted $\mathbf{M}^{\mathbf{F}^Q/O}$ and is defined in terms of $\mathbf{r}^{Q/O}$ (Q's position vector from O) as

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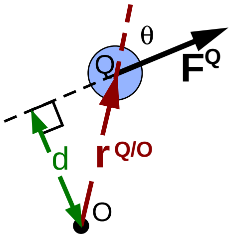
[a] A ***bound vector*** is a vector that is bound to a **point**.

$$\boxed{\mathbf{M}^{\mathbf{F}^Q/O} \triangleq \mathbf{r}^{Q/O} \times \mathbf{F}^Q} \qquad (1)$$

14.2.1 Moment arm of a vector about a point



The ***moment arm*** of the bound vector $\mathbf{F}^Q$ about point O is the distance d between O and the ***line of action***[a] of $\mathbf{F}^Q$ and measures the effectiveness of $\mathbf{F}^Q$ at creating a moment about O. This distance can be calculated in various ways, e.g, using the unit vector $\mathbf{u}$ in the direction of $\mathbf{F}^Q$ and/or the angle θ between $\mathbf{u}$ and $\mathbf{r}^{Q/O}$.

$$\boxed{d = \left|\mathbf{r}^{Q/O}\right| \sin(\theta) = \left|\mathbf{r}^{Q/O} \times \mathbf{u}\right| = \sqrt{\mathbf{r}^{Q/O} \cdot \mathbf{r}^{Q/O} - (\mathbf{r}^{Q/O} \cdot \mathbf{u})^2}}$$

$$\boxed{\left|\mathbf{M}^{\mathbf{F}^Q/O}\right| = \left|\mathbf{F}^Q\right| d}$$

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[a] The ***line of action*** of the ***bound vector*** $\mathbf{F}^Q$ is the line passing through Q and parallel to $\mathbf{F}^Q$.

14.2.2 Moment of a set of vectors

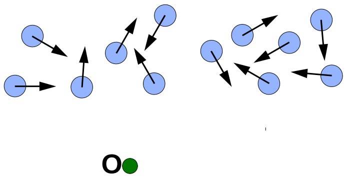
The ***moment*** of a set S of bound vectors $\mathbf{F}^{Q_1}, ..., \mathbf{F}^{Q_n}$ about a point O is defined as the sum of the moments of each bound vector in equation (2).

$$\boxed{\mathbf{M}^{S/O} \triangleq \sum_{i=1}^{n} \mathbf{M}^{\mathbf{F}^{Q_i}/O}} \qquad (2)$$

In view of Newton's law of action/reaction in Section 13.5.5, the moment of a system S's ***internal forces*** is $\mathbf{0}$ and $\mathbf{M}^{S/O}$ can be written **much more simply** as solely the ***moment of external forces*** on S.

$$\boxed{\mathbf{M}^{S/O} \underset{(13.7)}{=} \left(\mathbf{M}^{S/O}\right)_{\text{external}}} \qquad (3)$$

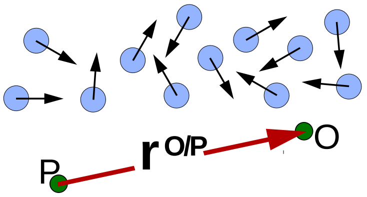
14.2.3 Statics, dynamics, and the moment of a set of forces



The moment $\mathbf{M}^{S/O}$ of **all** forces on a system S about an arbitrary point O is related to the time-derivative of S's angular momentum about O in N and other quantities. This relationships simplifies to ***static equilibrium*** in equation (4) when the system S is at ***rest*** (not moving) in N or S is considered massless.

$$\boxed{\mathbf{M}^{S/O} \underset{\text{(statics)}}{=} \mathbf{0}} \qquad (4)$$

14.2.4 Shift theorem for the moment of a set of vectors



$\mathbf{M}^{S/P}$, the moment of a set S of bound vectors about a point P, can be calculated in terms of:

- $\mathbf{M}^{S/O}$, the moment of S about a point O
- $\mathbf{r}^{O/P}$, O's position vector from P
- $\mathbf{F}^{S}$, the resultant of S

$$\boxed{\mathbf{M}^{S/P} \;=\; \mathbf{M}^{S/O} \;+\; \mathbf{r}^{O/P} \times \mathbf{F}^{S}} \qquad (5)$$

14.2.5 Torque of a set of vectors

Torque is the moment of a set S of vectors whose resultant is zero.

$$\boxed{\mathbf{T}^{S} \triangleq \mathbf{M}^{S/O} \quad \text{where} \quad \mathbf{F}^{S} = \mathbf{0} \quad \text{and point } O \text{ is } \textit{any} \text{ point}} \qquad (6)$$

Since a ***couple*** is a set of vectors whose resultant (sum) is $\mathbf{0}$, a ***torque*** is the ***moment of a couple***.[1] A couple has a special property, namely, the moment of a couple about a point O is equal to the moment of the couple about ***any*** other point Q. As a result, a ***torque is not associated with a point***.

The following example highlights the difference between a torque and a moment. Consider the various sets S of forces in Figure **??**, and fill in the following table by calculating $\mathbf{F}^{S}$, the resultant of S, and $\mathbf{M}^{S/O}$, $\mathbf{M}^{S/P}$, $\mathbf{M}^{S/Q}$, the moments of S about points O, P, and Q, respectively. Express your results in terms of the right-handed orthogonal unit vectors $\mathbf{n}_{\mathrm{x}}$, $\mathbf{n}_{\mathrm{y}}$, $\mathbf{n}_{\mathrm{z}}$.

S	$\mathbf{F}^{S}$	$\mathbf{M}^{S/O}$	$\mathbf{M}^{S/P}$	$\mathbf{M}^{S/Q}$	$\mathbf{M}^{S/O} = \mathbf{M}^{S/P} = \mathbf{M}^{S/Q}$?	Moment is a torque?
A	$10\,\mathbf{n}_{\mathrm{y}}$	$50\,\mathbf{n}_{\mathrm{z}}$	$\mathbf{0}$		**Yes/No**	**Yes/No**
B					**Yes/No**	**Yes/No**
C					**Yes/No**	**Yes/No**
D					**Yes/No**	**Yes/No**

14.2.6 Torque on a rigid body (or reference frame)

Torques can be associated with reference frames when the vectors that comprise the torque have a special character. The notation $\mathbf{T}^{A}$ is used to designate a torque of a couple whose vectors $\mathbf{F}_1, ..., \mathbf{F}_{\mathrm{n}}$ have lines of actions that pass through points $Q_1, ..., Q_n$, respectively, each of which is ***fixed*** in a reference frame A. When $Q_1, ..., Q_n$ are also ***fixed*** in a second reference frame, e.g., B, then one designates the torque $\mathbf{T}^{A/B}$.

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[1] Torques are a special type of Moment, namely the moment of a *couple*. All Torques are Moments, but not all Moments are Torques. A useful analogy is all Toyotas are Motor-vehicles, but not all Motor-vehicles are Toyotas.

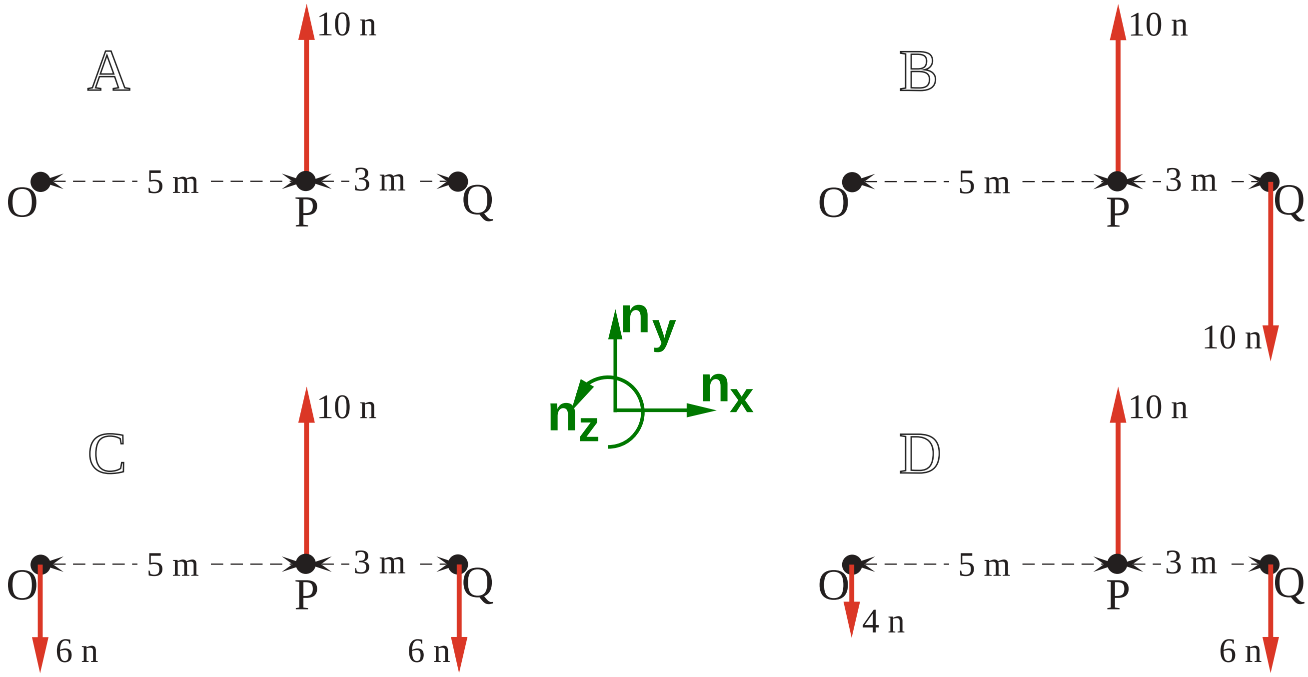

Figure 14.1: Various sets of forces

14.3 Proof of shift theorem for the moment of a set of bound vectors

To establish the validity of equation (5), consider a set S of bound vectors $\mathbf{F}_1, ..., \mathbf{F}_{\text{n}}$, whose lines of action pass through points Q_i $(i = 1, ..., n)$, respectively. The moment of S about a point P is defined in terms of $\mathbf{r}^{Q_i/P}$ (Q_i's position vector from P) as

$$\mathbf{M}^{S/P} \underset{(2)}{\triangleq} \sum_{i=1}^{n} \mathbf{r}^{Q_i/P} \times \mathbf{F}^{Q_i} \tag{7}$$

Q_i's position vector from P may be written as

$$\mathbf{r}^{Q_i/P} = \mathbf{r}^{O/P} + \mathbf{r}^{Q_i/O} \tag{8}$$

Substituting equation (8) into equation (7), distributing the summation, and rearrangement yields

$$\begin{aligned} \mathbf{M}^{S/P} &\underset{(7,8)}{=} \sum_{i=1}^{n} \mathbf{r}^{O/P} \times \mathbf{F}^{Q_i} + \sum_{i=1}^{n} \mathbf{r}^{Q_i/O} \times \mathbf{F}^{Q_i} \\ &= \mathbf{r}^{O/P} \times \sum_{i=1}^{n} \mathbf{F}^{Q_i} + \sum_{i=1}^{n} \mathbf{r}^{Q_i/O} \times \mathbf{F}^{Q_i} \end{aligned} \tag{9}$$

The first summation in equation (9) is the definition of $\mathbf{F}^S$ (the resultant of S). The second summation in equation (9) is the definition of $\mathbf{M}^{S/O}$ (the moment of S about O). Combining these two facts produces equation (5), the shift theorem for the moment of a set of bound vectors, namely

$$\mathbf{M}^{S/P} = \mathbf{r}^{O/P} \times \mathbf{F}^S + \mathbf{M}^{S/O}$$