OCT 21, 2009

ASSIGNMENT

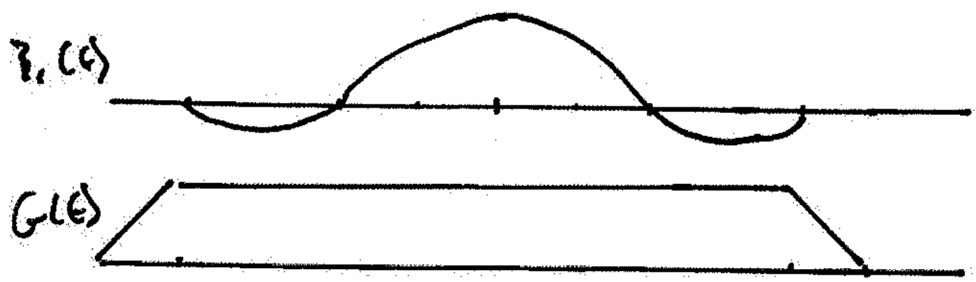
READ "PULSE DESIGN" p 53-58

TODAY

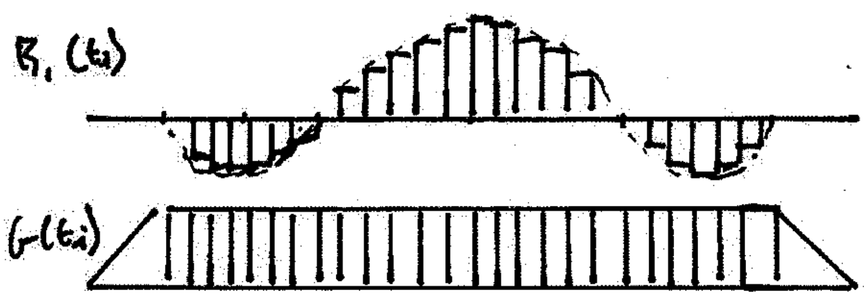
SHINNAR-LE ROUX TRANSFORM

CALCULATING RESPONSE OF LARGE-TIP-ANGLE PULSES

CONTINUOUS, LARGE-TIP-ANGLE PULSE



MODEL AS DISCRETE RECTANGLES



THE i^{th} RECTANGLE PRODUCES A ROTATION

$$\theta_i = -\gamma \Delta t \sqrt{|B_1(t_i)|^2 + (Gx)^2}$$

ABOUT AN AXIS

$$\underline{n}_i = \frac{\gamma \Delta t}{|\theta_i|} \left(B_{1,x}(t_i),\ B_{1,y}(t_i),\ Gx \right)$$

WE CAN THE COMPUTE

$$\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} \cos\theta_i/2 - i n_z \sin\theta_i/2 \\ -i(n_{x,i} + i n_{y,i}) \sin\theta_i/2 \end{pmatrix}$$

AND THE ROTATION MATRIX AS

$$Q_i = \begin{pmatrix} a_i & -b_i^* \\ b_i & a_i^* \end{pmatrix}$$

THE TOTAL ROTATION IS THEN

$$\underbrace{Q}_{\begin{pmatrix} \alpha_n & -\beta_n^* \\ \beta_n & \alpha_n^* \end{pmatrix}} = Q_n Q_{n-1} \ldots \underbrace{Q_i \ldots Q_2 Q_1}_{\begin{pmatrix} \alpha_i & -\beta_i^* \\ \beta_i & \alpha_i^* \end{pmatrix}}$$

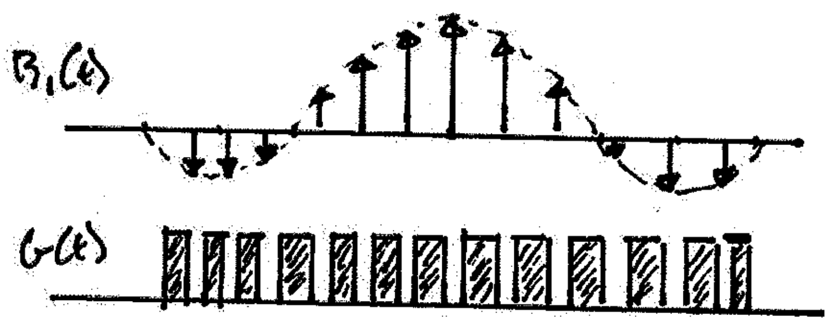
WE CAN COMPUTE THIS RECURSIVELY AS

$$\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} a_i & -b_i^* \\ b_i & a_i^* \end{pmatrix} \begin{pmatrix} \alpha_{i-1} \\ \beta_{i-1} \end{pmatrix}$$

SIMPLE TO COMPUTE, BUT LIMITED ANALYTICAL USE.

HARD PULSE APPROXIMATION

TREMENDOUS SIMPLIFICATION IF WE ASSUME RF CONSISTS OF IMPULSES SEPARATED BY FREE PRECESSION INTERVALS



GOOD APPROXIMATION TO CONTINUOUS "SOFT" PULSE IF ROTATIONS ARE SMALL

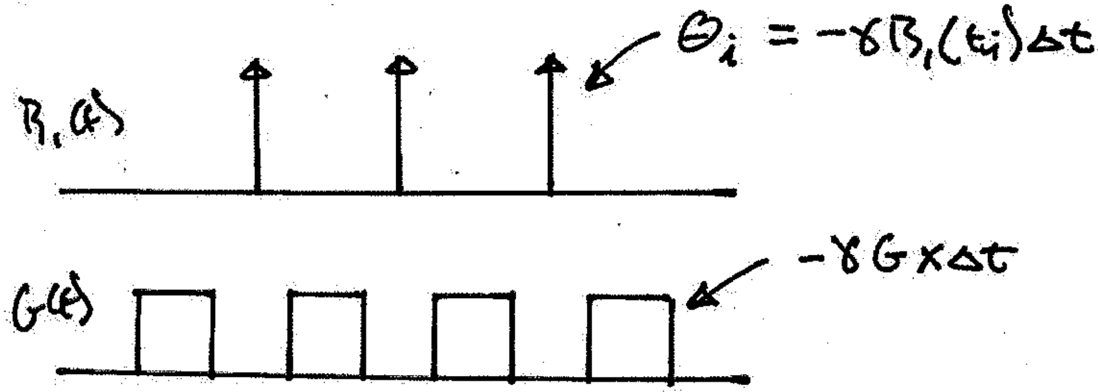
ARBITRARILY GOOD IF YOU SAMPLE FINELY ENOUGH

DUE TO

MEIR SHINNAR (+ JACK LEIGH)

PATRICK LE ROUX

ZOOMED IN VIEW



THE INCREMENTAL ROTATION MATRIX IS

$$Q_i = \underbrace{\begin{pmatrix} C_i & -S_i^* \\ S_i & C_i \end{pmatrix}}_{\text{HARD PULSE ROTATION}} \underbrace{\begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix}}_{\text{FREE PRECESSION}}$$

WHERE

$$C_i = \cos(\gamma |B_1(t_i)| \Delta t/2)$$

$$S_i = i \underbrace{e^{i\angle B_1(t_i)}}_{(n_x + i n_y)} \overbrace{\sin(\gamma |B_1(t_i)| \Delta t/2)}^{n_z = 0}$$

$$z = e^{i\gamma G x \Delta t}$$

IF WE SUBSTITUTE INTO THE RECURSION

$$\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} c_i & -s_i^* \\ s_i & c_i \end{pmatrix} \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \begin{pmatrix} \alpha_{i-1} \\ \beta_{i-1} \end{pmatrix}$$

$$= z^{1/2} \begin{pmatrix} c_i & -s_i^* \\ s_i & c_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} \alpha_{i-1} \\ \beta_{i-1} \end{pmatrix}$$

WE WANT TO GET RID OF HALF POWERS OF z, SO DEFINE

$$A_i = z^{-i/2} \alpha_i$$

$$B_i = z^{-i/2} \beta_i$$

THEN

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} = \begin{pmatrix} c_i & -s_i^* \\ s_i & c_i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} A_{i-1} \\ B_{i-1} \end{pmatrix}$$

THE INITIAL CONDITION IS NO ROTATION, SO

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

THEN

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} C_1 & -S_1^* \\ S_1 & C_1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$= \begin{pmatrix} C_1 \\ S_1 \end{pmatrix}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_2 & -S_2^* \\ S_2 & C_2 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} C_1 \\ S_1 \end{pmatrix}}_{\begin{pmatrix} C_1 \\ S_1 z^{-1} \end{pmatrix}}$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 C_2 - S_1 S_2^* z^{-1} \\ C_1 S_2 + S_1 C_2 z^{-1} \end{pmatrix}$$

POLYNOMIALS IN z^{-1}!

AT THE n^{th} TIME STEP

$$A_N(z) = \sum_{j=0}^{n-1} A_{Nj} z^{-j}$$

$$B_N(z) = \sum_{j=0}^{n-1} B_{Nj} z^{-j}$$

TWO $(n-1)$ ORDER POLYNOMIALS IN $z = e^{i\delta G x \delta t}$.

<u>FORWARD SLR TRANSFORM</u>

FAST RF PULSE SIMULATOR

1) EVALUATE THE COEFFICIENTS $\{A_{N,j}\}$ AND $\{B_{N,j}\}$ RECURSIVELY

SAME CALCULATION AS A SIMULATION OF A SINGLE POINT IN PROFILE

2) EVALUATE

$$A_N(z) = \sum_{j=0}^{N-1} A_{N,j} z^{-j}$$

$$B_N(z) = \sum_{j=0}^{N-1} B_{N,j} z^{-j}$$

ALONG THE UNIT CIRCLE $z = e^{i\gamma G x \Delta t}$ BY TAKING THE DFT OF $\{A_N\}$ AND $\{B_N\}$, WHICH CORRESPONDS TO EVALUATING THE PROFILE AT

$$x = n\Delta x = \frac{n}{\left(\frac{\gamma}{2\pi} G T\right)}$$

THEN

$$A_N(z) = \sum_{j=0}^{N-1} A_{N,j}\, e^{-i\gamma G\left(\frac{n}{\frac{\gamma}{2\pi}GT}\right)\Delta t\, j}$$

$$= \sum_{j=0}^{N-1} A_{N,j}\, e^{-i2\pi n \frac{1}{T}\frac{T}{N} j}$$

$$= \sum_{j=0}^{N-1} A_{N,j}\, e^{-i2\pi \frac{nj}{N}}$$

SIMILARLY

$$B_N(z) = \sum_{j=0}^{N-1} B_{N,j}\, e^{-i2\pi \frac{nj}{N}}$$

WHERE

$$z = e^{i\gamma G(n\Delta x)\Delta t}$$

3) COMPUTE DESIRED PROFILE

$$m_{xy}(n\Delta x) = Z\, A_N^*(z)\, B_N(z)\Big|_{z=e^{i\gamma G(n\Delta x)\Delta t}}$$

INVERSE SLR TRANSFORM

REMARKABLE FACT

GIVEN $A_N(z)$ AND $B_N(z)$, THE SLR TRANSFORM CAN BE INVERTED TO PRODUCE $B_1(t)$

$\Rightarrow$ IF WE CAN DESIGN $A_N(z)$ AND $B_N(z)$ WE CAN DESIGN $B_1(t)$.

MAGNITUDE CONSTRAINT

$$|A_N(z)|^2 + |B_N(z)|^2 = 1 \qquad z = e^{i\gamma G x \Delta t}$$

$(A_N(z), B_N(z))^T$ MUST BE A VALID ROTATION FOR ANY $|z| = 1$

BACK RECURSION

ONE STEP OF THE FORWARD SLR TRANSFORM

$$\underbrace{\begin{pmatrix} A_j \\ B_j \end{pmatrix}}_{\substack{j-1 \text{ ORDER} \\ \text{POLYNOMIALS}}} = \underbrace{\begin{pmatrix} C_j & -S_j^* z^{-1} \\ S_j & C_j z^{-1} \end{pmatrix}}_{\substack{\text{UNITARY} \\ \text{MATRIX} \\ Q_j}} \underbrace{\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix}}_{\substack{j-2 \text{ ORDER} \\ \text{POLYNOMIALS}}}$$

INVERSE RECURSION

$$\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix} = \underbrace{\begin{pmatrix} C_j & S_j^* \\ -S_j z & C_j z \end{pmatrix}}_{Q_j^* = Q_j^{-1}} \begin{pmatrix} A_j' \\ B_j' \end{pmatrix}$$

$$\begin{pmatrix} A_{j-1} \\ B_{j-1} \end{pmatrix} = \begin{pmatrix} C_j A_j' + S_j^* B_j' \\ z(-S_j A_j' + C_j B_j') \end{pmatrix}$$

WE KNOW $(A_j', B_j')^T$ AT EACH STAGE OF THE BACK RECURSION

ALSO, WE KNOW $(A_{j-1}, B_{j-1})^T$ ARE LOWER ORDER THAN $(A_j', B_j')^T$

$\Rightarrow$ LEADING TERM OF A_{j-1}' MUST DROP OUT

$\Rightarrow$ TRAILING TERM OF B_{j-1} MUST DROP OUT

$$C_j A_{j,j-1} + S_j^* B_{j,j-1} = 0$$ LEADING COEFFICIENTS

$$-S_j' A_{j,0} + C_j B_{j,0} = 0$$ TRAILING COEFFICIENTS

APPEAR TO BE TWO INDEPENDENT CONDITIONS, BUT ARE IN FACT THE SAME. FROM THE MAGNITUDE CONSTRAINT

$$|A_N(z)|^2 + |B_N(z)|^2 = 1$$

WE CAN SHOW THAT

$$A_{j,j-1} A_{j,0}^* + B_{j,j-1} B_{j,0}^* = 0$$

WITH THIS, EITHER OF THE CONSTRAINTS CAN BE DERIVED FROM THE OTHER.

CHOOSING THE LOW ORDER RELATION

$$-S_j A_{j,0} + C_j B_{j,0} = 0$$

$$S_j A_{j,0} = C_j B_{j,0}$$

$$\frac{B_{j,0}}{A_{j,0}} = \frac{S_j}{C_j}$$

$$= \frac{i e^{i\phi_j} \sin \theta_j/2}{\cos \theta_j/2}$$

$$= i e^{i\phi_j} \tan \theta_j/2$$

THEN

$$\theta_j = 2 \tan^{-1}\left(\left| \frac{B_{j,0}}{A_{j,0}} \right| \right)$$

$$\phi_j = \angle\left(-i\, B_{j,0}/A_{j,0} \right)$$

THE RF WAVEFORM IS

$$\underline{B_1(t_j) = \frac{1}{\gamma \Delta t} \theta_j e^{i\phi_j}}$$

INVERSE SLR TRANSFORM

SLR TRANSFORM

INVERTABLE RELATION BETWEEN

$$B_1(t) \overset{SLR}{\Longleftrightarrow} \left(A_N(z), B_N(z) \right)$$

SAME STRUCTURE TURNS UP IN MANY OTHER PLACES

"LAYER PEELING" ALGORITHMS

WAVE PROPOGATION THROUGH INHOMOGENEOUS MEDIA (SEISMOLOGY)

QMF FILTERS

QUADRATURE MIRROR FILTERS

PERFECT MULTIBAND DECIMATION AND RECONSTRUCTION FILTERS

LATTICE FILTERS

SPECIAL CASE OF A LATTICE FILTER WHERE EACH STAGE IS A EUCLIDEAN ROTATION.

GOOD QUANTIZATION AND DYNAMIC RANGE PROPERTIES