EE469B: Assignment 5 Solutions

Due Wednesday, Nov. 1

1. Spinor Magnitude Constraint Starting from the expressions for α and β as a function of a rotation angle θ about a unit vector $\mathbf{n} = (n_x, n_y, n_z)$, show that

$$\alpha\alpha^* + \beta\beta^* = 1$$

Solution:
The expressions for α and β are

$$\begin{aligned}\alpha &= \cos(\theta/2) - in_z \sin(\theta/2)\\ \beta &= -i(n_x + in_y)\sin(\theta/2).\end{aligned}$$

Then

$$\begin{aligned}\alpha\alpha^* &= (\cos(\theta/2) - in_z\sin(\theta/2))\,(\cos(\theta/2) + in_z\sin(\theta/2))\\ &== \cos^2(\theta/2) + n_z^2\sin^2(\theta/2)\end{aligned}$$

and

$$\begin{aligned}\beta\beta^* &= (-i(nx + in_y)\sin(\theta/2))\,(i(n_x - in_y)\sin(\theta/2))\\ &= (n_x^2 + n_y^2)\sin^2(\theta/2).\end{aligned}$$

Combining these two expressions

$$\begin{aligned}\alpha\alpha^* + \beta\beta^* &= \cos^2(\theta/2) + n_z^2\sin^2(\theta/2) + (n_x^2 + n_y^2)\sin^2(\theta/2)\\ &= \cos^2(\theta/2) + (n_z^2 + n_x^2 + n_y^2)\sin^2(\theta/2)\\ &= \cos^2(\theta/2) + \sin^2(\theta/2)\\ &= 1.\end{aligned}$$

2. Composite Pulses Often combinations of pulses are used to perform specific tasks more accurately. One example is the sequence consisting of a rotation of θ about the x axis, followed by a rotation by the same θ about the y axis. Our goal in this case is to start from equilbirum $\mathbf{M} = (0, 0, M_0)$ and to accurately make M_z zero. The magnetization is left somewhere in the transverse plane, but we don't care where.

a) Solve for α and β for this pulse sequence.

Solution
The rotation matrix for a θ rotation about x is

$$Q_1 = \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

and about y

$$Q_2 = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}.$$

Then α and β for the composite rotation is

$$\begin{aligned}
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} &= Q_2 Q_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ -i\sin\theta/2 \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta/2 + i\sin^2\theta/2 \\ \sin\theta/2\cos\theta/2 - i\sin\theta/2\cos\theta/2 \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta/2 + i\sin^2\theta/2 \\ (\sin\theta/2\cos\theta/2)(1-i) \end{pmatrix}
\end{aligned}$$

b) Find a simple expression for the longitudinal magnetization. Solve for the θ that produces an $M_z = 0$. Plot the M_z as a function of θ for $0 < \theta < \pi$. What range of θ will result in M_z being less than $0.05 M_0$?

Solution
The longitudinal magnetization is

$$\begin{aligned}
M_z &= 1 - 2\left|\beta\right|^2 \\
&= 1 - 2\left|(\sin\theta/2\cos\theta/2)(1-i)\right|^2 \\
&= 1 - 2\left|\frac{1}{2}\sin\theta(1-i)\right|^2 \\
&= 1 - \frac{1}{2}\sin^2\theta(\sqrt{2})^2 \\
&= 1 - \sin^2\theta \\
&= \cos^2\theta
\end{aligned}$$

This is plotted in Fig. 1 from 0 to 180 degrees, top, and 60 to 120 degrees bottom, to show that M_z is less than 5% of M_0 for $77^\circ < \theta < 103^\circ$.

c) Find a simple expression for the transverse magnetization. Plot M_{xy} as a function of θ for $0 < \theta < \pi$.

Solution
The transverse magnetization is

$$\begin{aligned}
M_{xy} &= 2\alpha^*\beta \\
&= 2\left(\cos^2\theta/2 - i\sin^2\theta/2\right)\left((1-i)\sin\theta/2\cos\theta/2\right) \\
&= (2\sin\theta/2\cos\theta/2)\left(\cos^2\theta/2 - i\sin^2\theta/2\right)(1-i)
\end{aligned}$$

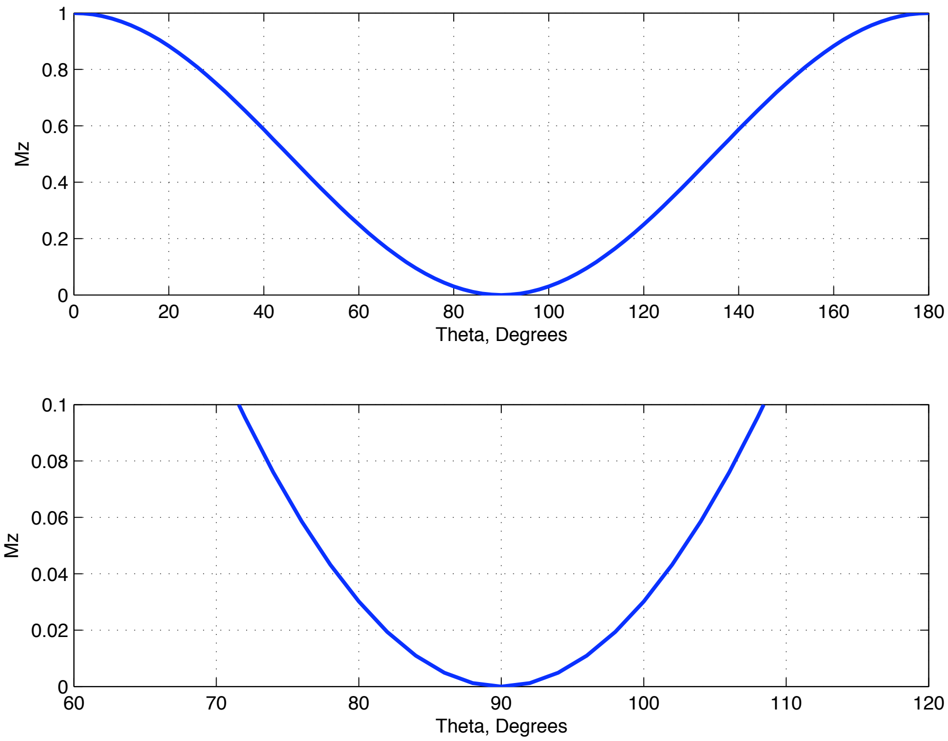

Figure 1: Plot of M_z for a $\theta_y\theta_x$ composite pulse, as a function of θ for $0 < \theta < 180$ above, and $60 < \theta < 120$ below.

$$
\begin{aligned}
&= \sin\theta\left(\cos^2\theta/2 - \sin^2\theta/2 - i\sin^2\theta/2 - i\cos^2\theta/2\right)\\
&= \sin\theta\left(\cos^2\theta/2 - \sin^2\theta/2 - i(\sin^2\theta/2 + \cos^2\theta/2)\right)\\
&= \sin\theta\,(\cos\theta - i)
\end{aligned}
$$

This is plotted in Fig. 2

d) Plot $\angle M_{xy}$ as a function of θ for $0 < \theta < \pi$. Note that this function is approximately linear about $\theta = \pi/2$. Hence, what this composite pulse does is trade off accuracy in M_z for phase error in M_{xy}. One use for this pulse sequence is the measurement of the RF field strength, since the actual flip angle θ_a is approximately the negative of magnetization phase $-\angle M_{xy}$. How far can θ_a vary from $\pi/2$ before the error in this approximation is 5°?.

Solution; The plot of $\angle M_{xy}$ and $-\theta$ are given in Fig. 3, above. In the lower plot, the error is shown. The error is within 5° for $55^\circ < \theta < 125^\circ$.

Hint: For these problems the half and double angle formulas are useful

$$
\begin{aligned}
\sin\theta &= 2\sin(\theta/2)\cos(\theta/2)\\
\cos\theta &= \cos^2(\theta/2) - \sin^2(\theta/2)
\end{aligned}
$$

Your answers shouldn't have any half angles.

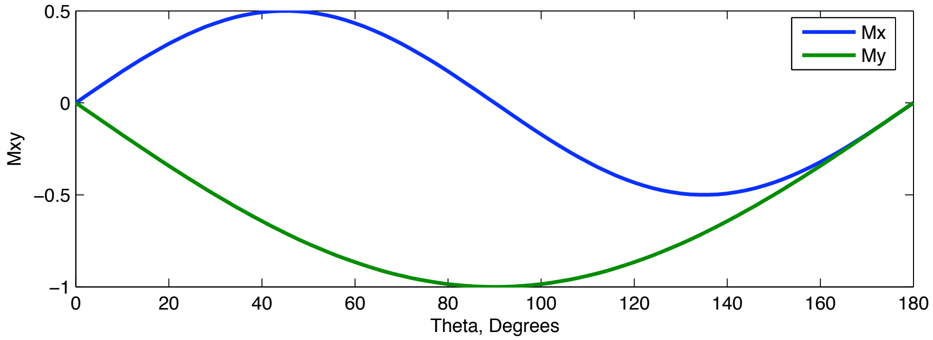

Figure 2: Plot of M_{xy} for a $\theta_y\theta_x$ composite pulse, as a function of θ for $0 < \theta < 180$.

3. Inverse SLR Transform On the web site are several m-files that implement the inverse SLR transform. These are

`b2a.m, mag2mp.m, ab2rf.m`

The first is `b2a.m`. This takes a beta polynomial and returns the consistent, minimum phase, alpha polynomial. It uses `mag2mp.m` to go from the magnitude profile of alpha to the minimum phase alpha. Then `ab2rf.m` takes the alpha and beta polynomials, and returns the corresponding RF pulse. There is also an m-file,

`ab2inv.m`

that takes the output of the simulator `abrm.m`, and returns the M_z profile that is produced.

Given a suitably scaled beta polynomial (i.e. the passband amplitude is $\sin(\theta/2)$), then the RF pulse can be computed as

```
>> a = b2a(b);
>> RF = ab2rf(a,b);
```

One way to think about the inverse SLR transform is that it corrects for the nonlinearity of the Bloch Equation. We can design the beta polynomial using Fourier arguments, just as a small tip angle pulse. Then scale the beta polynomial to $\sin(\theta/2)$, and apply the inverse SLR transform to produce the RF pulse with that beta profile.

a) Design a TBW=8 windowed sinc RF pulse, and scale it to a π radian rotation. Plot the RF pulse scaled to Gauss. Assume a pulse length of 8 ms, and a gradient strength of 0.425 G/cm. Simlate and plot the inversion profile. Choose an interesting range of spatial locations, and compute the inversion profile with

```
>> mz = ab2inv(abrm(rf,x));
```

b) Now, use the windowed sinc waveform as the beta polynomial. First, scale it to the proper value for an inversion. Then find the corresponding minimum phase and power alpha polynomial using `b2a.m`. Next compute the RF pulse using `ab2rf.m`. Plot the SLR inversion pulse and the windowed sinc inversion pulse from part (a), both scaled to Gauss. By what factor has the peak amplitude increased?

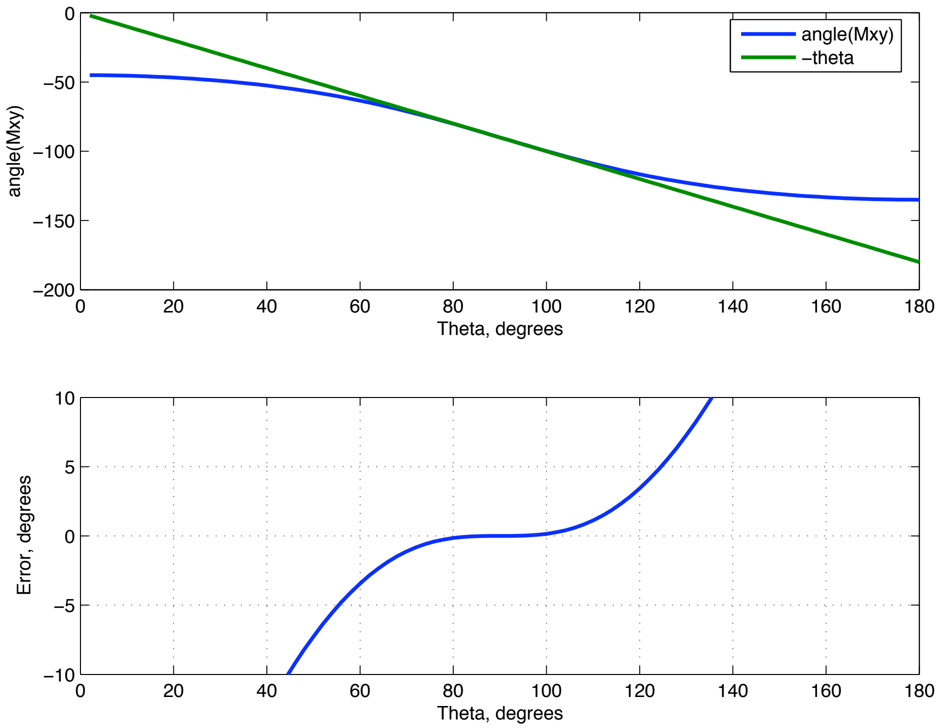

Figure 3: Plot of $\angle M_{xy}$ for a $\theta_y\theta_x$ composite pulse, as a function of θ for $0 < \theta < 180$. Note that this is approximately linear in the vicinity of $\theta = 90°$. The error in this approximation is plotted below.

c) Simulate the inversion profile of the SLR inversion pulse from part (b), and plot it along with the inversion profile of the windowed sinc pulse of part (a).

Solution:
The Fourier designed inversion pulse and the corresponding SLR inversion pulse are plotted in Fig. 4. The SLR design has almost twice the peak amplitude (1.9 times, to be precise), almost four times the peak power, and twice the integrated power. However, it is much more selective. It also reaches the expected slice width, since TBW = 8, and T=8, so BW = 1 kHz. A 0.425 G/cm gradient is 2 kHz/cm, so a 1 kHz bandwidth corresponds to 0.5 cm, which is what we observe for the half amplitude width in the slice profile plot. The Fourier design is considerably narrower.

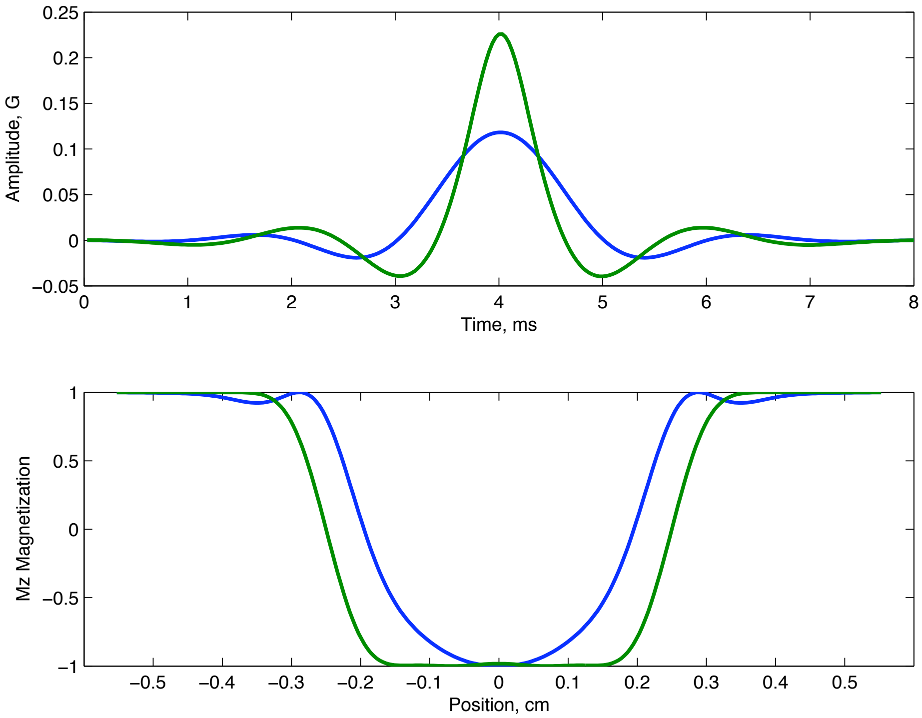

Figure 4: Comparison of a Fourier inversion pulse (top, blue) and a corresponding SLR design (top, green), and their inversion profiles (bottom).