Stanford University
Department of Electrical Engineering

EE 384X, Winter 2008
Solutions to Problem Set No: 2

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1. Recurrent, transient, positive and null states.

(a) $f_{ij}^{(n)}$ shows the probability of the event that if one starts from state i, one reaches the state j *for the first time* in n steps/jumps. In other words, it is the probability that the first visit to state j occurs at nth step, given that $X_0 = i$ (starting from state i). For $i = j$ a visit is called a *return*.

(b) f_{ij} is the probability that the chain *ever* visits state j, given $X_0 = i$. If $f_{ij} = 1$ it means that if the chain starts from state i, it will *definitely* visit state j in some step in future.

(c) If $f_{ii} = 1$ then state i is called *recurrent*, i.e., if the chain starts from state i, it will definitely return to state i. If $f_{ii} < 1$ then state i is called *transient*, i.e., it is possible that once the chain leaves state i, it never returns to it.

(d) T_{ij} is the number of steps that it takes to go to state j from state i for the first time.

(e) Using the properties of matrix P in this case:

$$f_{00}^{(n)} = a \quad \text{and} \quad f_{00}^{(n)} = 0 : \ \forall\, n > 1$$

$$f_{23}^{(1)} = 1 \quad \text{and} \quad f_{23}^{(n)} = 0 : \ \forall\, n > 1$$

$$f_{01}^{(n)} = a^{n-1} b$$

Hence:

$$f_{00} = a \ , \ f_{11} = 1 \ , \ f_{22} = 1 \ , \ f_{33} = 1$$

So states $1, 2, 3$ are recurrent and state 0 is transient when $a < 1$.

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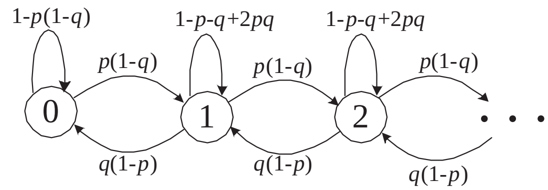
2. Finite Birth-Death Chain.

(a) The birth-death chain is shown in this figure:

Let $p' = p(1-q)$ and $q' = q(1-p)$. p' shows the probability that we have an arrival but no departure (so queue size increases) and q' shows the probability of that a



departure occurs but no arrival (queue size decreases). Also let $\rho = \frac{p'}{q'} = \frac{p(1-q)}{q(1-p)}$. Note that if $p < q$, then $p' < q'$ and therefore $\rho < 1$. From the Markov chain we get the following set of balance equations:

$$\begin{cases} p'\pi_0 = q'\pi_1 & \Rightarrow \quad \pi_1 = \rho\pi_0 \\ p'\pi_1 = q'\pi_2 & \Rightarrow \quad \pi_2 = \rho\pi_1 = \rho^2\pi_0 \\ p'\pi_2 = q'\pi_3 & \Rightarrow \quad \pi_3 = \rho\pi_2 = \rho^3\pi_0 \\ \vdots & \qquad \vdots \\ p'\pi_{n-1} = q'\pi_n & \Rightarrow \quad \pi_n = \rho\pi_{n-1} = \rho^n\pi_0 \\ \vdots & \qquad \vdots \end{cases}$$

So $\pi_n = \rho^n \pi_0$, and to find π_0 note that (assuming $\rho < 1$)

$$\sum_{n=0}^{\infty} \pi_n = 1 \quad \Rightarrow \quad \pi_0 \sum_{n=0}^{\infty} \rho^n = 1 \quad \Rightarrow \quad \frac{\pi_0}{1-\rho} = 1 \quad \Rightarrow \quad \pi_0 = 1 - \rho.$$

Hence,

$$\pi_n = (1-\rho)\rho^n \quad n = 0, 1, 2, \ldots$$

(b)

$$\mathbb{E}(Q) = \sum_{n=0}^{\infty} n\pi_n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n$$

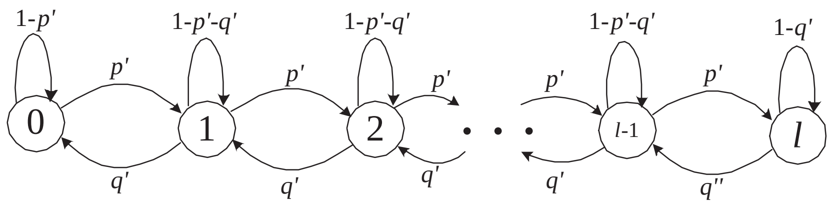
Note that $\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho}$, differentiating both sides with respect to ρ, we get

$$\sum_{n=0}^{\infty} n\rho^{n-1} = \frac{1}{(1-\rho)^2}.$$

Hence,

$$\mathbb{E}(Q) = (1-\rho)\rho\sum_{n=0}^{\infty} n\rho^{n-1} = \frac{\rho}{1-\rho}.$$

If $q < p$ then $q' < p'$ and the queue is not stable, so $\mathbb{E}(Q) = \infty$. (All states in the chain are transient).



(c) The Markov chain is shown in the following figure:

We can interpret the behavior of system when the buffer is full, in two different ways.

- **First Case:** When the queue is full (l customers are in the line), all new arrivals are lost even if a customer leaves at that time slot.
- **Second Case:** If queue is full, a new customer is admitted only if a customer leaves at that time slot.

In the first case, the probability of going from state l to state $l-1$ is given by $q'' = q$, while in second case this probability is $q'' = q(1-p) = q'$. Define $\alpha = \frac{q'}{q''}$ as

$$\alpha = \begin{cases} \frac{q(1-p)}{q} = 1-p & \text{first case} \\ 1 & \text{second case} \end{cases}$$

The balance equations are as follows:

$$\begin{cases} p'\pi_0 = q'\pi_1 & \Rightarrow \quad \pi_1 = \rho\pi_0 \\ p'\pi_1 = q'\pi_2 & \Rightarrow \quad \pi_2 = \rho\pi_1 = \rho^2\pi_0 \\ p'\pi_2 = q'\pi_3 & \Rightarrow \quad \pi_3 = \rho\pi_2 = \rho^3\pi_0 \\ \vdots & \qquad\quad \vdots \\ p'\pi_{l-2} = q'\pi_{l-1} & \Rightarrow \quad \pi_{l-1} = \rho\pi_{l-1} = \rho^{l-1}\pi_0 \\ p'\pi_{l-1} = q''\pi_l & \Rightarrow \quad \pi_l = \alpha\rho\pi_{l-1} = \alpha\rho^l\pi_0 \end{cases}$$

Hence,

$$\pi_n = \begin{cases} \rho^n\pi_0 & 0 \le n \le l-1, \\ \alpha\rho^l\pi_0 & n = l, \\ 0 & n > l. \end{cases}$$

To find π_0, note that:

$$\sum_{n=0}^{\infty}\pi_n = 1 \quad \Rightarrow \quad \sum_{n=0}^{l-1}\pi_0\rho^n + \alpha\rho^l\pi_0 = 1 \quad \Rightarrow \quad \pi_0\left(\frac{1-\rho^l}{1-\rho} + \alpha\rho^l\right) = 1.$$

So

$$\pi_0 = \frac{1}{\frac{1-\rho^l}{1-\rho} + \alpha\rho^l}.$$

(d)

$$\mathbb{E}(Q) = \sum_{n=0}^{l} n\pi_0\rho^n + \alpha\pi_0 l\rho^l$$

To find the above sum, note that $\sum_{n=0}^{l-1}\rho^n = \frac{1-\rho^l}{1-\rho}$, by differentiating both sides with respect to ρ , we get:

$$\sum_{n=0}^{l-1} n\rho^{n-1} = \frac{-l\rho^{l-1}(1-\rho)+(1-\rho^l)}{(1-\rho)^2} = \frac{1+(l-1)\rho^l - l\rho^{l-1}}{(1-\rho)^2}.$$

Hence,

$$\mathbb{E}(Q) = \pi_0\rho\frac{1+(l-1)\rho^l - l\rho^{l-1}}{(1-\rho)^2} + \alpha\pi_0 l\rho^l.$$

All the above results are valid for $q \geq p$ (unlike part (a)). Observe that if $p = q$, then $p' = q'$, then from the balance equations, we get the following distribution:

$$\pi_n = \begin{cases} \frac{1}{l+\alpha} & 0 \leq n \leq l-1, \\ \frac{\alpha}{1+\alpha} & n = l, \\ 0 & n > l. \end{cases}$$

Note that for $\alpha = 1$ (second case), the distribution is uniform.

(e) Note that limit of π_0 (obtained in part (c)) as l goes to infinity is the same as π_0 at part (a).

$$\lim_{l\to\infty} \frac{1}{\frac{1-\rho^l}{1-\rho} + \alpha\rho^l} = 1-\rho.$$

Hence,

$$\begin{aligned}
\lim_{l\to\infty} \mathbb{E}(Q) &= \lim_{l\to\infty} \pi_0\rho\frac{1+(l-1)\rho^l - l\rho^{l-1}}{(1-\rho)^2} + \alpha\pi_0 l\rho^l \\
&= (1-\rho)\frac{\rho}{(1-\rho)^2} + 0 \\
&= \frac{\rho}{1-\rho},
\end{aligned}$$

which is the same as expression in part (b) for $\mathbb{E}(Q)$.

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3. Time reversibility of Markov Chains. (a) We assume that $X_0 \sim \pi$, so that

$X_n \sim \pi$, for n = 1, 2, ,N. We have:

$$\begin{aligned} P(Y_{n+1} = y|Y_n = x) &= P(X_{N-n-1} = y|X_{N-n} = x) \\ &= \frac{P(X_{N-n-1} = y, X_{N-n} = x)}{P(X_{N-n} = x)} \\ &= \frac{P(X_{N-n} = x|X_{N-n-1} = y)P(X_{N-n-1} = y)}{P(X_{N-n} = x)} \\ &= \frac{p(y,x)\pi(y)}{\pi(x)} \\ &= \frac{\pi(y)}{\pi(x)}p(y,x) \end{aligned}$$

(b) The detailed balance condition $p(x,y)\pi(x) = p(y,x)\pi(y)$, for all $x, y \in S$ is equivalent to $p(x,y) = p(y,x)\frac{\pi(y)}{\pi(x)} = P(Y_{n+1} = y|Y_n = x)$. By definition, $p(x,y) = P(X_{n+1} = y|X_n = x)$, so the Markov chain is time reversible if and only if π satisfies the detailed balance conditions for P.

(c) For the first matrix, the stationary distribution $\pi = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$ satisfies the detailed balance equation, as $(\frac{\beta}{\alpha+\beta}).\alpha = (\frac{\alpha}{\alpha+\beta}).\beta$. So the first one is a transition matrix for a time-reversible Markov chain. Being doubly stochastic, the second matrix has stationary distribution $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3},)$ which only satisfies the detailed balance condition if $p = \frac{1}{2}$. It is not time-reversible for any other values of p.

4. Random walks on Graphs (a) It follows from the definition of a graph (all points connected to at least one other point) that this system is irreducible. Since the state space is finite their exists a unique stationary distribution. Notice:

$$\begin{aligned} \pi(x)p(x,y) &= \frac{A(x,y)}{d(x)} \frac{d(x)}{\sum_{z\in S} d(z)} \\ &= \frac{A(x,y)}{\sum_{z\in S} d(z)} \frac{d(y)}{d(y)} \\ &= \frac{d(y)}{\sum_{z\in S} d(z)} \frac{A(x,y)}{d(y)} \\ &= \pi(y)p(y,x) \end{aligned}$$

since $A(x,y) = A(y,x)$. So we have satisfied the detailed balance equations and must have the unique stationary distribution.

(b)Note that $\sum_{z\in S} d(z) = 778 + 645 + 43 = 420$. There are 3 types of squares to consider for the king in the inner square the king can move 8 places, on an edge the king can move 5 places and in a corner the king can move 3 places. So the stationary distribution for each of those three types of places is 8/420, 5/420 and 3/420 respectively. Then the mean return time to the corner starting from the corner is just 1 over the stationary distribution probability of being in the corner which $420/3 = 140$ moves expected before returning to the corner.

(c)For the knight we have there are 4 corners with 2 places to move, 8 edges with 3 places and 16 edges with 4 places. 4 inner squares enable the knight to move 4 places and 16 inner squares allow for 6 places to move and 16 inner squares for 8 places to move, hence $\sum d(z) = 336$. The expected number of moves to return to the corner is then $336/2 = 168$.

5. Symmetric Random Walk. (a) Note that for odd n, $P_{00}^{(n)} = 0$ and for even n, we can calculate $P_{00}^{(n)}$ exactly as follows:

$$P_{00}^{(n)} = \frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)}(\frac{1}{2})^n$$

Using the stirling approximation for both factorial terms, it's easy to see that

$$P_{00}^{(n)} \approx \frac{C}{\sqrt{n}}$$

where C is a positive constant. Thus:

$$\sum_n P_{00}^{(n)} = \infty$$

which shows that state 0 and hence the whole chain is recurrent (since all states commute).

(b) The key is that X_n and Y_n are simply independent symmetric random walks on the integers. So in this case:

$$P_{(0,0),(0,0)}^{(n)} = (\frac{n!}{(\frac{n}{2}!)(\frac{n}{2}!)}(\frac{1}{2})^n)^2$$

for even n. But by the same approximation as in (a):

$$P_{(0,0),(0,0)}^{(n)} \approx \frac{C}{n}$$

So once again we have

$$\sum_n P_{(0,0),(0,0)}^{(n)} = \infty$$

so the chain is recurrent.

(c) As before, X_n, Y_n, Z_n are now independent symmetric random walks on the integers, so by the same approximation as in (a) and (b):

$$P_{(0,0,0),(0,0,0)}^{(n)} \approx \frac{C}{n^{1.5}}$$

But now we have:

$$\sum_n P^{(n)}_{(0,0,0),(0,0,0)} < \infty$$

So the chain is transient in this case. Think about what this means. Intuitively, it seems three dimensional space is so big, that if you just wonder around randomly, you might get lost forever (never return to where you started), while part (b) implies that this never happens in two dimensions.

(d) The difference between this and part (b) is that we can no longer claim that X_n and Y_n are independent (for example think what the event $\{X_2 = X_1 + 1\}$ tells you about the Y_n process), so the same analysis can't be applied directly. But the thing to note is that if you rotate this walk 45 degrees, and scale it properly, it's identical to the walk in part (b). Another way to think about this is to consider the projections of the walk along the lines $y = x$ and $y = -x$. It's clear that each of these projections are now simple symmetric random walks. So as in (b) this chain must also be recurrent.

6. Non-Uniform Output Destinations.
(a) The three states of the Markov chain are:

(1) (2,0): HoL packets at both inputs want to go to output 1.

(2) (1,1): HoL packets at inputs want to go to outputs 1 and 2, respectively.

(3) (0,2): HoL packets at both inputs want to go to output 2.

Assume that when HoL packets at both inputs are destined for output 1, output 1 chooses input 1 with probability r_1 and input 2 with probability $r_2 = 1-r_1$. Similarly, output 2 chooses inputs 1 and 2 with probability λ_1 and λ_2. The transition probability matrix is

$$P = \begin{bmatrix} r_1p_{11} + r_2q_{21} & r_1p_{12} + r_2q_{22} & 0 \\ p_{11}q_{21} & 1 - p_{11}q_{21} - p_{12}q_{22} & p_{12}q_{22} \\ 0 & \lambda_1p_{11} + \lambda_2q_{21} & \lambda_1p_{12} + \lambda_2q_{22} \end{bmatrix}$$

The stationary distribution is

$$\pi = \left(\frac{\alpha}{1+\alpha+\beta}, \frac{1}{1+\alpha+\beta}, \frac{\beta}{1+\alpha+\beta}\right),$$

where

$$\begin{aligned} \alpha &= \frac{p_{11}q_{21}}{1 - r_1p_{11} - r_2q_{21}} \\ \beta &= \frac{p_{12}q_{22}}{1 - \lambda_1p_{12} - \lambda_2q_{22}}. \end{aligned}$$

Therefore, the switch throughput is

$$1.\frac{\alpha}{1+\alpha+\beta} + 2.\frac{1}{1+\alpha+\beta} + 1.\frac{\beta}{1+\alpha+\beta} = 1 + \frac{1}{1+\alpha+\beta}.$$

(b) When $r_1 = \lambda_1 = .5$ and $p_{11} = .4$ and $q_{21} = .6$, we get

$$\begin{aligned} \pi &= (0.2449, 0.5102, 0.2449) \\ \text{switch throughput} &= 1.51 \\ \text{per output throughput} &= .76 \end{aligned}$$

(c) When $r_1 = \lambda_1 = .5$ and $p_{11} = .2$ and $q_{21} = .8$, we get

$$\begin{aligned} \pi &= (0.1951, 0.6098, 0.1951) \\ \text{switch throughput} &= 1.61 \\ \text{per output throughput} &= .81 \end{aligned}$$

(d) Yes, we can adjust r_1 and λ_1 so that the throughput is improved. Throughput (equal to $1 + \frac{1}{1+\alpha+\beta}$) can be maximized by choosing r_1 to minimize α and λ_1 to minimize β. Since

$$\begin{aligned} \alpha &= \frac{p_{11}q_{21}}{1 - r_1 p_{11} - r_2 q_{21}} \\ &= \frac{p_{11}q_{21}}{1 - q_{21} + r_1(q_{21} - p_{11})}, \end{aligned}$$

choosing $r_1 = 1$ if $q_{21} > p_{11}$ and 0 else will minimize α. Similarly, β is minimized by choosing $\lambda_1 = 1$ if $q_{22} > p_{12}$ and 0 else.

Thus, when $p_{11} = 0.2$ and $q_{21} = 0.8$ the throughput is maximized by choosing $r_1 = 1$ and $\lambda_1 = 0$. The maximum switch throughput in this case is 1.71 and the maximum per output throughput is 0.86.

Similarly, when $p_{11} = 0.4$ and $q_{21} = 0.6$ choosing $r_1 = 1$ and $\lambda_1 = 0$ maximizes throughput. The maximum values are: switch throughput = 1.56 and the per output throughput = 0.78.

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7. Using MGF.

For $M/M/\infty$ we have the following set of equations:

$$\begin{cases} \lambda p_0 = \mu p_1 & \Rightarrow \quad \lambda p_0 = \mu p_1 \\ \lambda p_1 = 2\mu p_2 & \Rightarrow \quad \lambda p_1 z = 2\mu p_2 z \\ \lambda p_2 = 3\mu p_3 & \Rightarrow \quad \lambda p_2 z^2 = 3\mu p_3 z^2 \\ \vdots & \\ \lambda p_{n-1} = n\mu p_n & \Rightarrow \quad \lambda p_{n-1} z^{n-1} = n\mu p_n z^{n-1} \\ \vdots & \qquad \vdots \end{cases}$$

By adding the equations, we will get:

$$\lambda(p_0 + p_1 z + p_2 z^2 + \cdots) = \mu(p_1 + 2p_2 z + 3p_3 z^2 + \cdots)$$

Note that

$$\Phi(z) = \sum_{n=0}^{\infty} p_n z^n \Rightarrow \Phi'(z) = p_1 + 2p_2 z + 3p_3 z^2 + \cdots = \sum_{n=1}^{\infty} n p_n z^{n-1}$$

Thus

$$\lambda\Phi(z) = \mu\Phi'(z) \Rightarrow \Phi'(z) = \frac{\lambda}{\mu}\Phi(z)$$

The general solution of this differential equation has the form $\Phi(z) = Ke^{\frac{\lambda}{\mu}z}$ with K being and arbitrary constant. Using initial condition $\Phi(1) = 1$, we obtain:

$$K = e^{-\frac{\lambda}{\mu}}$$

Hence,

$$\Phi(z) = e^{\frac{\lambda}{\mu}(z-1)}$$

which is the MGF for Poisson random variable with mean $\frac{\lambda}{\mu}$, so:

$$p_n = e^{-(\frac{\lambda}{\mu})}\frac{(\frac{\lambda}{\mu})^n}{n!}.$$

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8. Max-Min Fairness.

To prove that definitions (a) and (b) are equivalent, we need to show that for any given collection of N flows $f_1 \leq \ldots \leq f_N$ the assignment generated by (b) satisfies the definition (a) and vice versa. Recall that definition (a) was the abstract definition of max-min fairness, while definition (b) was the algorithmic definition (as discussed in class).

Proof that the solution of (b) satisfies (a): Let the rates obtained by definition (b), that is the algorithm, be $(r_1, \ldots, r_N)$ for demands $(f_1, \ldots, f_N)$, with the given property $f_i \leq f_j$ for $i \leq j$. Note that, the algorithm in (b) assigns values to $r_1, \ldots, r_N$ in that order, one-by-one. It makes a binary decision every time: i.e. whether to assign $r_i = f_i$(denote it by 0) or $r_i = \frac{R}{N-i}$(denote this by 1), the equal share of the remaining normalized capacity. Thus, the execution of algorithm can be denoted by a string of length N consisting of 0s and 1s. We claim that, the possible execution of the algorithm corresponds to $0-1$ strings of the form $0^k 1^{N-k}$, $0 \leq k \leq N$. That is, the first k smallest flows obtain rates equal to their demand, and rest of the flows obtain equal rates. Thus, the only way to increase any of the rates while maintaining feasibility are to increase these equal flows, and to do so, the flows that decrease in value are shorter or equal to its value. Hence it satisfies definition (a).

Proof that the solution of (a) satisfies (b): If the rate assignment $r = (r_1, \ldots, r_N)$ satisfies the definition (a) then the following is true:

(o) If $\sum_i f_i \leq C$ then for all i, $r_i = f_i$. If not, then there must exist a flow i such that $r_i < f_i$ and for any other flow j, the rate assigned $r_j \leq f_j$ due to the feasibility condition. Thus $\sum_i r_i < C$. Let $\epsilon = C - \sum_i r_i > 0$. Hence we can increase the rate r_i by an amount ϵ without decreasing any other r_j. But this is a contradiction to the definition (a).
(i) For all flows $USAT = \{i : r_i < f_i\}$, the rates assigned are same. If not then there exist i, j such that $r_i < f_i, r_j < f_j$, and $r_i < r_j$. In that case, we can increase r_i by $\epsilon > 0$ and decrease r_j by ϵ and still remain feasible; which contradicts definition (a).
(ii) If for a flow i, $r_i = f_i$ then all flows $1, \ldots, i-1$ having lower demands then flow i must be assigned the rates as $r_k = f_k, 1 \leq k \leq i-1$. If not, then one can argue in a similar vein to (i) and obtain a contradiction to (a).
(iii) If the smallest demand $f_1 < C/N$, then it must be satisfied. If not, then by (o), it must be the case that $\sum_i f_i > C$. Hence at least one of the flows, say j is assigned a rate $r_j \geq C/N$. We can decrease r_j by $\epsilon > 0$ and increase r_1 by ϵ; but again this is contradicts definition (a).
(iv) Using (o)-(iii) inductively we obtain the algorithm described in definition (b). Thus, definition (a) $\Rightarrow$ definition (b).

Instead of showing that, definition (a) $\Rightarrow$ definition (b), one can prove that, there is a unique solution that satisfies definition (a). This will, in turn, imply that it has to be the solution obtained by algorithm of definition (b).

We have presented the above longer solution to show that, given an abstract definition, it is possible to obtain a constructive solution. This might be helpful when *a priori* it is not known what constructive solution should be.

Thus we have shown the equivalence of the two definitions (a) and (b).

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9. (σ, ρ) **Constrained Traffic.**
(a) *Case $\rho > C$:* Traffic arrives at higher rate than service rate. Hence queue can grow in unbounded fashion.
Case $\rho \leq C$: Initially $Q(0) = 0$, and for any t before it gets empty again, the following is true:

$$\begin{aligned} Q(t) &= A(0,t) - Ct \\ &\leq \rho t + \sigma - Ct \\ &\overset{(i)}{\leq} \sigma \end{aligned}$$

where (i) is true as $\rho \leq C$. Thus, maximum backlog is σ.

(b) *Case $\rho > C$:* Again, queue can grow in unbounded fashion.
Case $\rho = C$: For this case, queue size can grow in unbounded fashion.

Case $\rho < C$: Similar to previous case, we obtain,

$$\begin{aligned} Q(t) &= A(0,t) - Ct \\ &\leq \rho t + \rho\sqrt{t} + \sigma - Ct \end{aligned} \tag{1}$$

The Equation (1) is maximized for

$$t^* = \left(\frac{\rho}{2(C-\rho)}\right)^2.$$

and the maximum backlog is

$$Q(t^*) = \frac{\rho^2}{4(C-\rho)} + \sigma.$$

(c)
(i): Consider the unique integer k such that $k\tau \leq t < (k+1)\tau$. Then, using the periodicity property of A and the fact that $t - k\tau \geq 0$, we get:

$$A(t, t+\tau) = A(t-k\tau, t+\tau-k\tau) = A(t-k\tau, t-(k-1)\tau).$$

By the same periodicity property, we also have:

$$A(\tau, t-(k-1)\tau) = A(0, t-k\tau).$$

Therefore, after cutting $A(t-k\tau, t-(k-1)\tau)$ into two parts, one gets:

$$A(t,t+\tau) = A(t-k\tau,\tau) + A(\tau, t-(k-1)\tau) = A(t-k\tau,\tau) + A(0,t-k\tau) = A(0,\tau) = A.$$

(ii): Let $0 \leq s < t$. Problem 1.c.(i) shows that $A(s,t) = A(0, t-s)$. Consider now the smallest integer n such that $t - s \leq n\tau$, i.e. $n = \lceil \frac{t-s}{\tau} \rceil$. Then $A(s,t) \leq A(0, n\tau) = nA(0,\tau) = nA$.

Using the well-known inequality $\lceil x \rceil < x+1$, it is easy to derive the following inequality: $A(s,t) < (\frac{t-s}{\tau} + 1)A = \frac{A}{\tau}(t-s) + A$. Comparing with the definition of (σ, ρ), we thus get $\rho = \frac{A}{\tau}$ and $\sigma = A$.

(iii): Using Problems 1.(a) and 1.c.(ii), the maximum backlog is $\sigma = A$. It is achieved for instance with a periodic traffic of period τ such that there is an arrival burst of size σ at time $t = 0$, and then no arrival from 0 to τ.

(d)
(i): The queue at time t, $Q(t)$, increases due to $a(t+1)$ arrivals and decreases due to at most C departures. If after arrivals, queue has no more than C customers, then it becomes zero. This gives the Lindley's equation,

$$Q(t+1) = \max(Q(t) + a(t+1) - C, 0),$$

(ii): We want to determine $Q(t)$. If $Q(t) = 0$ then note that for any $s \leq t$,

$$A(t) - A(s) - C(t - s) \leq 0.$$

In particular, for $s = t$,

$$A(t) - A(s) - C(t - s) = 0,$$

which proves desired statement.

If $Q(t) \neq 0$, then let $s^* = \max\{0 \leq s \leq t : Q(s) = 0\}$. Note that $Q(0) = 0$ and hence such an s^* exists. By definition,

$$Q(s) > 0, \forall s^* < s \leq t.$$

Using this fact along with inductive application of (i) gives,

$$Q(t) \quad = \quad A(t) - A(s^*) - C(t - s^*) \tag{2}$$

We first claim the following:

Claim:

$$A(t) - A(s^*) - C(t - s^*) > A(t) - A(s) - C(t - s), \quad \forall s^* < s \leq t.$$

Proof:
Note that, for any $s > s^*$, from (i) we obtain,

$$\begin{aligned} Q(t) &= A(t) - A(s) - C(t - s) + Q(s) \\ &> A(t) - A(s) - C(t - s) \end{aligned} \tag{3}$$

¿From (2) and (3),

$$A(t) - A(s^*) - C(t - s^*) \quad > \quad A(t) - A(s) - C(t - s) \tag{4}$$

This proves the claim.

Next, consider any $0 \leq y \leq s^*$. Consider the following:

$$\begin{aligned} A(t) - A(y) - C(t - y) &= A(t) - A(s^*) + A(s^*) - A(y) - C(t - s^*) - C(s^* - y) \\ &= [A(t) - A(s^*) - C(t - s^*)] + [A(s^*) - A(y) - C(s^* - y)] \\ &\overset{(x)}{\leq} [A(t) - A(s^*) - C(t - s^*)] \end{aligned} \tag{5}$$

where (x) is true as $Q(s^*) = 0$, and hence the arrival happened in time (y, s^*) should be no more than service $C(s^* - y)$.

¿From (2) and (5),

$$Q(t) = \max_{0 \leq s \leq t} [A(t) - A(s) - C(t - s)].$$

(iii) Given $Q(t) \leq B$, we obtain from (ii)

$$Q(t) \quad = \quad \max_{0 \leq s \leq t} [A(t) - A(s) - C(t-s)] \leq B \tag{6}$$

¿From (6), for all s, t we obtain:

$$A(t) - A(s) - C(t-s) \leq B,$$

that is,

$$A(t) - A(s) = A(s,t) \leq B + C(t-s).$$

Thus parameters are $\rho = C, \sigma = B$.

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10. Packetized Fair Queueing and Deficit Round Robin

a)
(i) Packetized bit-by-bit fair queueing: A1, A2, C1, B1, A3, B2, C2, C3, A4, B3.
(ii) Deficit round robin with quantum size Q=1 bit: same as (i)
(iii) Deficit round robin with Q=3 bits: A1, A2, C1, A3, B1, B2, C2, C3, A4, B3

b) Note that the deficit round robin server is work conserving because decisions can be made in no time and server is busy as long as there is work.
(i) Packetized bin-by-bit fair queueing: The new packet is shceduled to depart, in round 2, before all the other HoL packets in queue 1 through queue $N-1$, which should depart in round a. But the HoL packet of queue 1 is being served now, so the new packet has to be served after it. So the new packet will be served at time a, and depart at time $a+1$.
(ii) Deficit round robin with quantum size $Q = 1$ bit: At time zero, the scanner will place all the $N-1$ HoL packets into the output FIFO. So the new packet can only be served after the output FIFO becomes empty, at time $a(N-1)$. It will depart at time $a(N-1)+1$.