# Note regarding a question asked by Martin Morf (EE380, 2004/10/13)

The question was "Why is the following Mathematica formula incorrect?".

$$\sum_{i=1}^{n} \sum_{j=i}^{m} 1 \quad = \quad \frac{n \cdot (2 \cdot m - n + 1)}{2} \qquad (0)$$

Since summation using $\sum$ is ubiquitous in mathematics, it is worthwhile to elaborate. We consider two interpretations of the question:

a. Why must equation (0) be considered incorrect?

b. How can we explain the design error in Mathematica?

The short answers were in the material projected during the presentation (page 8):

a. For $n := 3$ and $m := 1$ the r.h.s. of (0) is 0 whereas $\sum_{i=1}^{3} \sum_{j=i}^{1} 1 = 1$.

b. The rules for $\sum$ are not as well-understood as commonly assumed.

A more complete answer (which could not be fitted within the time span available at the colloquium) is given here. Regarding b., we provide a "reverse engineering" of (0).

$$\begin{aligned}
\textstyle\sum_{i=1}^{n} \sum_{j=i}^{m} 1 \quad &= \quad \langle \textstyle\sum_{k=a}^{b} c = (b+1-a)\cdot c \rangle \quad && \textstyle\sum_{i=1}^{n} m + 1 - i \\
&= \quad \langle \text{Distributivity of } \textstyle\sum / - \rangle && (\textstyle\sum_{i=1}^{n} m + 1) - (\textstyle\sum_{i=1}^{n} i) \\
&= \quad \langle \textstyle\sum_{k=a}^{b} c = (b+1-a)\cdot c \rangle && n \cdot (m+1) - (\textstyle\sum_{i=1}^{n} i) \\
&= \quad \langle \textstyle\sum_{k=a}^{b} k = (b+1-a)\cdot \frac{a+b}{2} \rangle && n \cdot (m + 1 - \tfrac{n+1}{2}) \\
&= \quad \langle \text{Rewriting by arithmetic} \rangle && n \cdot (2 \cdot m - n + 1)/2
\end{aligned}$$

The error resides in blindly using rules like $\sum_{k=a}^{b} c = (b+1-a)\cdot c$ even if $b < a$.

In our formalism (Funmath), we avoid such errors by properly defining $\sum$ and deriving a collection of formal calculation rules. For the nomenclature and various symbols used next, we refer to the pdf file of the presentation used on October 13.

We define $\sum$ as an elastic operator with 3 axioms: for any $a$, any numeric $c$ and any number-valued functions $f$ and $g$ with nonintersecting finite domains, we introduce

0. the *empty rule*: $\sum \varepsilon = 0$ ;
1. the *one-point rule*: $\sum (a \mapsto c) = c$ ;
2. the *merge rule*: $\sum (f \cup g) = \sum f + \sum g$ .

From these basic rules, all rules needed in practice are derived. Noteworthy is the *trading rule* $\sum f_P = \sum (P \,\widehat{?}\, f)$, which is similar to the rule $\forall\, Q_P = \forall\, (P \,\widehat{\Rightarrow}\, Q)$ and even more to $\exists\, Q_P = \exists\, (P \,\widehat{\wedge}\, Q)$ for quantifiers (see the presentation material page 28).

As usual, taking abstractions yields pointwise expressions and formal rules for them. In particular, we define $\sum_{k=a}^{b} e$ as standing for $\sum k : a\,..\,b\,.\,e$ for integer $a$ and $b$ and numeric $e$, where the set $a\,..\,b$ is defined as in PASCAL by $a\,..\,b = \{k : \mathbb{Z} \mid a \leq k \leq b\}$. Note that in case $b < a$ one has $a\,..\,b = \emptyset$ and then $\sum_{k=a}^{b} e = \sum k : \emptyset\,.\,e = \sum \varepsilon = 0$.

Formal calculation (with the trading rule in the star role) yields the correct formula

$$\sum_{i=1}^{n} \sum_{j=i}^{m} 1 \quad = \quad (k \geq 1)\,?\, \frac{k \cdot (2 \cdot m - k + 1)}{2} \ \dagger\ 0 \ \textbf{where}\ k := \min(m, n) \qquad (1)$$

for any integer $n$ and $m$ (exercise), which provides the complete answer regarding a.