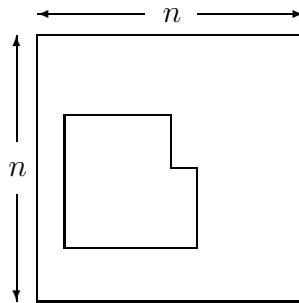


Solutions to Final Exam

Each problem is worth 20 points.

1. Kolmogorov complexity



What is the Kolmogorov complexity (to first order) of a *square* with a *rectangle* eaten out of a corner? The square and rectangle are on an $n \times n$ grid with the axes lined up with the borders.

Solution: Kolmogorov complexity

This question is similar to question 1f of homework set 6. To describe the square, one needs $2 \log(n)$ bits to describe the upper left hand corner, and $\log(n)$ bits to specify the length of an edge. We can then describe the rectangle by first describing the corner of the square it intersects, this takes $\log(4)$ bits, and then specify the opposing corner of the rectangle, which takes $2 \log(n)$ bits.

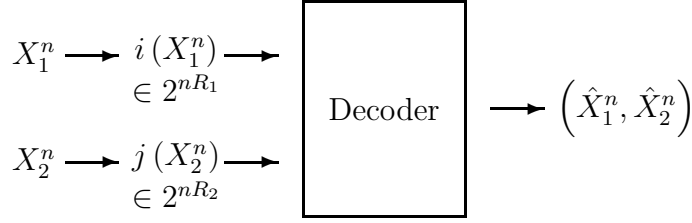
$$K(\text{square with a rectangle eaten out of a corner} | n) \leq 5 \log(n) + c.$$

2. Slepian Wolf

Let

$$\begin{aligned} X_1 &= U \oplus Z_1, \\ X_2 &= U \oplus Z_2, \end{aligned}$$

where $U \sim \text{Bern}(p)$, $Z_1 \sim \text{Bern}(\alpha_1)$, and $Z_2 \sim \text{Bern}(\alpha_2)$, and let U , Z_1 , and Z_2 be independent. What rates (R_1, R_2) suffice to describe X_1 and X_2 ?



Solution: Slepian Wolf

We know

$$\begin{aligned}
 R_1 &\geq H(X_1|X_2) \\
 R_2 &\geq H(X_2|X_1) \\
 R_1 + R_2 &\geq H(X_1, X_2).
 \end{aligned}$$

$$H(X_1, X_2) = H(p\alpha_1\alpha_2 + \bar{p}\bar{\alpha}_1\bar{\alpha}_2, p\alpha_1\bar{\alpha}_2 + \bar{p}\bar{\alpha}_1\alpha_2, p\bar{\alpha}_1\alpha_2 + \bar{p}\alpha_1\bar{\alpha}_2, p\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}\alpha_1\alpha_2).$$

$$\begin{aligned}
 H(X_1|X_2) &= H(X_1, X_2) - H(X_2) \\
 &= H(p\alpha_1\alpha_2 + \bar{p}\bar{\alpha}_1\bar{\alpha}_2, p\alpha_1\bar{\alpha}_2 + \bar{p}\bar{\alpha}_1\alpha_2, p\bar{\alpha}_1\alpha_2 + \bar{p}\alpha_1\bar{\alpha}_2, p\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}\alpha_1\alpha_2) \\
 &\quad - H_B(\alpha_2 * p)
 \end{aligned}$$

where H_B is the binary entropy function and $\alpha_2 * p = \bar{p}\alpha_2 + p\bar{\alpha}_2$. Similarly,

$$\begin{aligned}
 H(X_2|X_1) &= H(X_1, X_2) - H(X_1) \\
 &= H(p\alpha_1\alpha_2 + \bar{p}\bar{\alpha}_1\bar{\alpha}_2, p\alpha_1\bar{\alpha}_2 + \bar{p}\bar{\alpha}_1\alpha_2, p\bar{\alpha}_1\alpha_2 + \bar{p}\alpha_1\bar{\alpha}_2, p\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}\alpha_1\alpha_2) \\
 &\quad - H_B(\alpha_1 * p).
 \end{aligned}$$

Plugging back into the Slepian-Wolf region gives

$$\begin{aligned}
 R_1 &\geq H(p\alpha_1\alpha_2 + \bar{p}\bar{\alpha}_1\bar{\alpha}_2, p\alpha_1\bar{\alpha}_2 + \bar{p}\bar{\alpha}_1\alpha_2, p\bar{\alpha}_1\alpha_2 + \bar{p}\alpha_1\bar{\alpha}_2, p\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}\alpha_1\alpha_2) \\
 &\quad - H_B(\alpha_2 * p) \\
 R_2 &\geq H(p\alpha_1\alpha_2 + \bar{p}\bar{\alpha}_1\bar{\alpha}_2, p\alpha_1\bar{\alpha}_2 + \bar{p}\bar{\alpha}_1\alpha_2, p\bar{\alpha}_1\alpha_2 + \bar{p}\alpha_1\bar{\alpha}_2, p\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}\alpha_1\alpha_2) \\
 &\quad - H_B(\alpha_1 * p) \\
 R_1 + R_2 &\geq H(p\alpha_1\alpha_2 + \bar{p}\bar{\alpha}_1\bar{\alpha}_2, p\alpha_1\bar{\alpha}_2 + \bar{p}\bar{\alpha}_1\alpha_2, p\bar{\alpha}_1\alpha_2 + \bar{p}\alpha_1\bar{\alpha}_2, p\bar{\alpha}_1\bar{\alpha}_2 + \bar{p}\alpha_1\alpha_2).
 \end{aligned}$$

3. Constrained signals

What is the channel capacity of a binary symmetric channel with crossover probability $p = 3/4$ and “energy” constraint $EX^2 \leq 1/3$, $\mathcal{X} = \{0, 1\}$?

Solution: Constrained signal

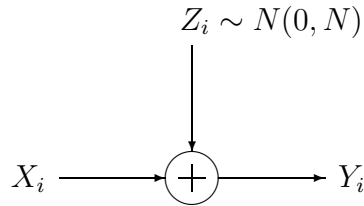
Since $X_i \in \{0, 1\}$, the only parameter at our disposal is $\beta = \Pr\{X_i = 1\}$.

$$\begin{aligned} I(X; Y) &= H(Y) - H(Y|X) \\ &= H(Y) - H_B(3/4) \\ &= H_B(\beta * 3/4) - H_B(1/4). \end{aligned}$$

Therefore

$$\begin{aligned} C &= \sup_{P_X} I(X; Y) \\ &= \sup_{\beta \in [0, 1/3]} H_B(\beta * 3/4) - H_B(1/4) \\ &= H_B(1/3 * 3/4) - H_B(1/4) \\ &= H_B(5/12) - H_B(1/4). \end{aligned}$$

4. Gaussian channel



$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n X_i(W)^2 &\leq P \\ \frac{1}{n} \sum_{i=1}^n X_i(W) &\geq \alpha, \quad \text{for all } W \in 2^{nR}. \end{aligned}$$

What is the capacity of this additive Gaussian noise channel under the two constraints? What distribution on X achieves the capacity? The answer should be in closed form.

Solution: Gaussian channel

We first examine the case where $\alpha > 0$.

$$\begin{aligned}
 C &= \sup_{\{P_X: EX \geq \alpha, EX^2 \leq P\}} I(X; Y) \\
 &= \sup_{\{P_X: EX \geq \alpha, EX^2 \leq P\}} h(Y) - h(Y|X) \\
 &= \sup_{\{P_X: EX \geq \alpha, EX^2 \leq P\}} h(Y) - h(Z) \\
 &= \sup_{\{P_X: EX \geq \alpha, EX^2 \leq P\}} h(Y) - \frac{1}{2} \log(2\pi eN).
 \end{aligned}$$

Since X and Z are independent, the variance of $Y = X + Z$ is the sum of the variances which is bounded by $P + N - \alpha^2$. Hence

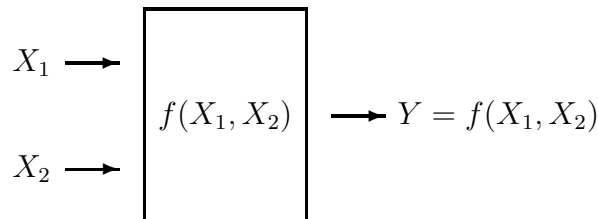
$$h(Y) \leq \frac{1}{2} \log(2\pi e(P + N - \alpha^2)).$$

The the upper bound is achieved when $X \sim N(\alpha, P - \alpha^2)$. Therefore

$$\begin{aligned}
 C &= \frac{1}{2} \log(2\pi e(P + N - \alpha^2)) - \frac{1}{2} \log(2\pi eN) \\
 &= \frac{1}{2} \log \left(1 + \frac{P - \alpha^2}{N} \right).
 \end{aligned}$$

When $\alpha \leq 0$ we can take $X \sim N(0, P)$. Therefore,

$$C = \begin{cases} \frac{1}{2} \log \left(1 + \frac{P - \alpha^2}{N} \right) & \text{if } \alpha > 0 \\ \frac{1}{2} \log \left(1 + \frac{P}{N} \right) & \text{if } \alpha \leq 0. \end{cases}$$

5. Multiple access channel

What is the capacity region of the deterministic multiple access channel if $f(X_1, X_2)$ is invertible, where $X_1 \in \{1, 2, \dots, m_1\}$ and $X_2 \in \{1, 2, \dots, m_2\}$?

Solution: Multiple access channel

For all deterministic MAC's the capacity region simplifies to

$$\begin{aligned} R_1 &\leq I(X_1; Y|X_2) = H(Y|X_2) \\ R_2 &\leq I(X_2; Y|X_1) = H(Y|X_1) \\ R_1 + R_2 &\leq I(X_1, X_2; Y) = H(Y). \end{aligned}$$

Since $f(X_1, X_2)$ is invertible, then given Y we know (X_1, X_2) . Therefore

$$\begin{aligned} R_1 &\leq H(Y|X_2) = H(X_1) \\ R_2 &\leq H(Y|X_1) = H(X_2) \\ R_1 + R_2 &\leq H(Y) = H(X_1, X_2). \end{aligned}$$

These inequalities are simultaneously maximize by picking uniform distributions over the alphabets $\{1, 2, \dots, m_1\}$ and $\{1, 2, \dots, m_2\}$. This gives the following capacity region

$$\begin{aligned} R_1 &\leq \log(m_1) \\ R_2 &\leq \log(m_2) \\ R_1 + R_2 &\leq \log(m_1) + \log(m_2). \end{aligned}$$

6. Velocity distribution

Suppose we have atoms in a 10-dimensional box.

- (a) What is the maximum entropy distribution on the velocity vector $\mathbf{v} = (v_1, v_2, \dots, v_{10}) \in \mathbb{R}^{10}$ subject to the kinetic energy constraint

$$E \frac{1}{2} m \|\mathbf{V}\|^2 = E_o?$$

- (b) Suppose there is also a potential energy contribution. An atom at height $x \geq 0$ and velocity $\mathbf{v} \in \mathbb{R}^{10}$ has total energy $\frac{1}{2} m \|\mathbf{v}\|^2 + mgx$. Find the maximum entropy density $f(x, \mathbf{v})$ subject to the constraints $x \geq 0$ and

$$E \left(\frac{1}{2} m \|\mathbf{V}\|^2 + mgX \right) = E_o.$$

Solution: Velocity distribution

(a) From Theorem 11.1.1 of EIT, we know that the maximum entropy distribution is

$$f^*(\mathbf{v}) = C e^{-\lambda \frac{1}{2} m \|\mathbf{v}\|^2},$$

where λ is chosen to satisfy the constraint

$$E_{f^*} \frac{1}{2} m \|\mathbf{V}\|^2 = E_o$$

and C is a normalizing constant. Notice that f^* simplifies to a product distribution

$$f^*(\mathbf{v}) = C \prod_{i=1}^{10} e^{-\lambda \frac{1}{2} m v_i^2}$$

and $\mathbf{V}_i \sim N\left(0, \frac{1}{\lambda m}\right)$. Therefore

$$\begin{aligned} E_{f^*} \frac{1}{2} m \|\mathbf{V}\|^2 &= 10 \left(E \frac{1}{2} m \mathbf{V}_1^2 \right) \\ &= 10 \left(\frac{m}{2\lambda m} \right) \\ &= \frac{5}{\lambda}. \end{aligned}$$

Setting

$$E_{f^*} \frac{1}{2} m \|\mathbf{V}\|^2 = E_o$$

then gives $\lambda = 5/E_o$, $\mathbf{V}_i \sim N\left(0, \frac{1}{5m} E_o\right)$, and $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{10}$ are independent.

(b) Similarly to part (a),

$$f^*(\mathbf{v}, x) = C e^{-\lambda \frac{1}{2} m \|\mathbf{v}\|^2 - \lambda m g x},$$

where λ is chosen to satisfy the constraint

$$E_{f^*} \left(\frac{1}{2} m \|\mathbf{V}\|^2 + m g X \right) = E_o$$

and C is a normalizing constant. Once again, f^* simplifies to a product distribution

$$f^*(\mathbf{v}, x) = C e^{-\lambda m g x} \prod_{i=1}^{10} e^{-\lambda \frac{1}{2} m v_i^2}$$

and $\mathbf{V}_i \sim N\left(0, \frac{1}{\lambda m}\right)$ and X has the exponential distribution with parameter $\lambda m g$. Therefore

$$\begin{aligned} E_{f^*} \left(\frac{1}{2} m \|\mathbf{V}\|^2 + m g X \right) &= 10 \left(E \frac{1}{2} m \mathbf{V}_1^2 \right) + m g E X \\ &= 10 \left(\frac{m}{2\lambda m} \right) + m g \frac{1}{\lambda m g} \\ &= \frac{6}{\lambda}. \end{aligned}$$

Setting

$$E_{f^*} \left(\frac{1}{2} m \|\mathbf{V}\|^2 + mgX \right) = E_o$$

then gives $\lambda = 6/E_o$, $\mathbf{V}_i \sim N(0, \frac{1}{6m} E_o)$, X has the exponential distribution with parameter $\frac{6mg}{E_o}$, and $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{10}, X$ are independent. Therefore,

$$E_{f^*} \frac{1}{2} m \|\mathbf{V}\|^2 = \frac{10}{12} E_o.$$

7. Rate distortion

What is the rate distortion function $R(D)$ for a Bernoulli(p) source and distortion function

$$d(x, \hat{x}) = \begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{array}{cc} \left| \begin{array}{cc} 5 & 2 \\ 2 & 5 \end{array} \right. \end{array} \end{array}$$

Solution: Rate distortion

Without loss of generality, assume that $p \in [0, .5]$. Due to symmetry in the distortion function, we may limit ourselves to conditional distributions $\Pr\{X|\hat{X}\}$ with $\Pr\{X_i = 1|\hat{X}_i = 1\} = \Pr\{X_i = 0|\hat{X}_i = 0\}$. Let $\alpha = \Pr\{X_i = 1|\hat{X}_i = 1\} = \Pr\{X_i = 0|\hat{X}_i = 0\}$. Therefore

$$\begin{aligned} Ed(X, \hat{X}) &= 5\alpha + 2\bar{\alpha} \\ &= 3\alpha + 2. \end{aligned}$$

Hence $Ed(X, \hat{X}) \leq D$ if and only if

$$\alpha \leq \frac{D-2}{3}.$$

Expanding $I(X; \hat{X})$ we get

$$\begin{aligned} I(X; \hat{X}) &= H(X) - H(X|\hat{X}) \\ &= H_B(p) - H_B(\alpha), \end{aligned}$$

where H_B is the binary entropy function and $\bar{\alpha} * p = \bar{p}\bar{\alpha} + p\alpha$.

Given $D \in [2, 3p + 2]$,

$$\begin{aligned}
R(D) &= \min_{\{\Pr\{\hat{x}|x\}:Ed(X,\hat{X})\leq D\}} I(X; \hat{X}) \\
&= \min_{\alpha \in [0, (D-2)/3]} I(X; \hat{X}) \\
&= \min_{\alpha \in [0, (D-2)/3]} H_B(p) - H_B(\alpha) \\
&\stackrel{(a)}{=} H_B(p) - H_B\left(\frac{D-2}{3}\right) \quad \text{for all } D \in [2, 3p + 2], \tag{1}
\end{aligned}$$

where (a) follows from the fact that $H_B(p) - H_B(\alpha)$ is a decreasing function of α on the range $[0, p]$ and $D \in [2, 3p + 2]$. We can now look at $R(D)$ for general D .

$$\begin{aligned}
R(D) &= \min_{\{p(\hat{x}|x):Ed(X,\hat{X})\leq D\}} I(X; \hat{X}) \\
&\stackrel{(a)}{=} \begin{cases} \infty & \text{if } D < 2 \\ \min_{\{p(\hat{x}|x):Ed(X,\hat{X})\leq D\}} I(X; \hat{X}) & \text{if } D \in [2, 3p + 2] \\ \min_{\{p(\hat{x}|x):Ed(X,\hat{X})\leq D\}} I(X; \hat{X}) & \text{if } D > 3p + 2 \end{cases} \\
&\stackrel{(b)}{=} \begin{cases} \infty & \text{if } D < 2 \\ H_B(p) - H_B\left(\frac{D-2}{3}\right) & \text{if } D \in [2, 3p + 2] \\ \min_{\{p(\hat{x}|x):Ed(X,\hat{X})\leq D\}} I(X; \hat{X}) & \text{if } D > 3p + 2 \end{cases} \\
&\stackrel{(c)}{=} \begin{cases} \infty & \text{if } D < 2 \\ H_B(p) - H_B\left(\frac{D-2}{3}\right) & \text{if } D \in [2, 3p + 2] \\ 0 & \text{if } D > 3p + 2, \end{cases}
\end{aligned}$$

where (a) follows from the fact that $Ed(X, \hat{X}) \geq 2$, (b) from equation (1), and (c) follows from the fact that $R(D) \leq R(3p + 2)$ for all $D > 3p + 2$ and $R(3p + 2) = 0$.

8. Likelihood ratio

Let X_1, X_2, \dots be i.i.d. $\sim Q(x)$. Suppose one performs a likelihood ratio test of P_1 vs P_2 . However, neither is true.

(a) Find

$$\Pr \left\{ \sum_{i=1}^n \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha \right\}$$

if X_1, X_2, \dots are i.i.d. $\sim Q(x)$.

(b) Conditioned on

$$\sum_{i=1}^n \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha,$$

what distribution P^* does the data “look like?” Distinguish the two cases for Q :

$$\sum_x Q(x) \ln \frac{P_1(x)}{P_2(x)} \geq \alpha$$

and

$$\sum_x Q(x) \ln \frac{P_1(x)}{P_2(x)} \leq \alpha.$$

Solution: Likelihood ratio

(a) Using Sanov's theorem

$$\Pr \left\{ \sum_{i=1}^n \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha \right\} \doteq 2^{-nD(P^*||Q)},$$

where P^* minimizes $D(P||Q)$ subject to the constrain

$$E_P \ln \frac{P_1(X)}{P_2(X)} \geq \alpha.$$

If $E_Q \ln \frac{P_1(X)}{P_2(X)} \geq \alpha$ then $P^* = Q$ and $D(P^*||Q) = 0$. Therefore

$$\Pr \left\{ \sum_{i=1}^n \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha \right\} \doteq 1.$$

If $E_Q \ln \frac{P_1(X)}{P_2(X)} < \alpha$, then from equation 12.110 in EIT,

$$\begin{aligned} P^*(x) &= \frac{Q(x)e^{\lambda \ln \frac{P_1(x)}{P_2(x)}}}{\int_y Q(y)e^{\lambda \ln \frac{P_1(y)}{P_2(y)}} dy} \\ &= \frac{Q(x) \left(\frac{P_1(x)}{P_2(x)} \right)^\lambda}{\int_y Q(y) \left(\frac{P_1(y)}{P_2(y)} \right)^\lambda dy} \\ &= CQ(x)P_1^\lambda(x)P_2^{-\lambda}(x). \end{aligned}$$

Therefore,

$$\Pr \left\{ \sum_{i=1}^n \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha \right\} \doteq \begin{cases} 2^{-nD(P^*||Q)} = 2^{-n(\lambda\alpha + \ln C)} & \text{if } E_Q \ln \frac{P_1(X)}{P_2(X)} < \alpha \\ 1 & \text{if } E_Q \ln \frac{P_1(X)}{P_2(X)} \geq \alpha, \end{cases}$$

where

$$P^*(x) = CQ(x)P_1^\lambda(x)P_2^{-\lambda}(x).$$

(b) From the conditional limit theorem, we know that

$$\Pr \left\{ X_j \left| \sum_{i=1}^n \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha \right. \right\} \doteq \begin{cases} CQ(x)P_1^\lambda(x)P_2^{-\lambda}(x) & \text{if } E_Q \ln \frac{P_1(X)}{P_2(X)} < \alpha \\ Q(X_j) & \text{if } E_Q \ln \frac{P_1(X)}{P_2(X)} \geq \alpha. \end{cases}$$

9. Horse race

A horse race has win probabilities (p_1, p_2, \dots, p_m) . The bettor places bets (b_1, b_2, \dots, b_m) , where $b_i \geq 0$ and $\sum_{i=1}^m b_i = 1$. The odds-maker offers odds o_i for $1, i = 1, 2, \dots, m$, where the odds are consistent with some probability mass function (r_1, r_2, \dots, r_m) , with $r_i \geq 0$ and $\sum_{i=1}^m r_i = 1$, in the sense that $o_i = 1/r_i, i = 1, 2, \dots, m$. Let

$$W = \sum_{i=1}^m p_i \ln(b_i o_i)$$

be the growth rate of wealth. Let

$$v = \min_r \max_b W.$$

Find the minimax strategies b^* and r^* , and the value v of this game.

Solution: Horse race

Given odds o , let us first look at $\max_b W(o, b)$.

$$\begin{aligned}
0 &= \frac{\partial}{\partial b_i} \left(W + \lambda \sum_{j=1}^m b_j \right) \\
0 &= \frac{\partial}{\partial b_i} \left(\sum_{j=1}^m p_j \ln(b_j o_j) + \lambda \sum_j b_j \right) \\
0 &= \frac{p_i}{b_i o_i} o_i + \lambda \\
-\lambda &= \frac{p_i}{b_i} \\
-\lambda b_i &= p_i.
\end{aligned}$$

Since $\sum_{j=1}^m p_j = \sum_{j=1}^m b_j = 1$ and $p_i = -\lambda b_i$ for all i , then $\lambda = -1$ and $b_i^* = p_i$ for all i . Hence regardless of the odds o ,

$$\max_b W(o, b) = \sum_{i=1}^m p_i \ln(p_i o_i).$$

To minimize over o , let

$$\begin{aligned}
0 &= \frac{\partial}{\partial o_i} \left(\max_b W(o, b) + \lambda \sum_{j=1}^m \frac{1}{o_j} \right) \\
0 &= \frac{\partial}{\partial o_i} \left(\sum_{j=1}^m p_j \ln(p_j o_j) + \lambda \sum_{j=1}^m \frac{1}{o_j} \right) \\
0 &= \frac{p_i}{o_i p_i} p_i - \lambda \frac{1}{o_i^2} \\
-\lambda &= \frac{p_i}{o_i} \\
-\frac{\lambda}{o_i} &= p_i.
\end{aligned}$$

Since $\sum_{j=1}^m p_j = \sum_{j=1}^m 1/o_j = 1$ and $p_i = -\lambda/o_i$ for all i , then $\lambda = -1$ and $o_i^* = 1/p_i$ for all i .

Therefore, $r_i^* = b_i^* = p_i$ for all i , and $v = 0$.