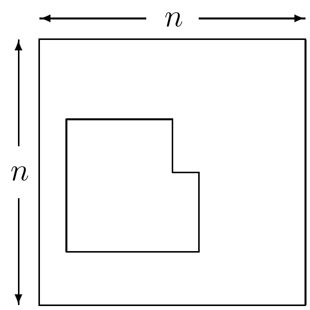
EE376B/Stat 376B
Information Theory
Prof. T. Cover

Handout #23
Monday, June 6, 2005

Final Exam

Each problem is worth 20 points.

1. **Kolmogorov complexity**

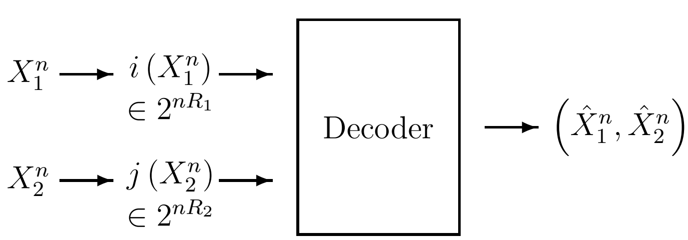


What is the Kolmogorov complexity (to first order) of a *square* with a *rectangle* eaten out of a corner? The square and rectangle are on an $n \times n$ grid with the axes lined up with the borders.

2. **Slepian Wolf**
Let

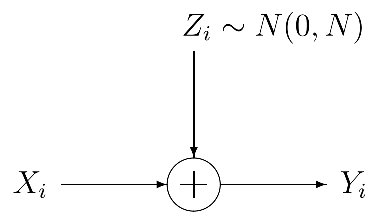
$$X_1 = U \oplus Z_1,$$
$$X_2 = U \oplus Z_2,$$

where $U \sim$Bern(p), $Z_1 \sim$Bern(α_1), and $Z_2 \sim$Bern(α_2), and let U, Z_1, and Z_2 be independent. What rates (R_1, R_2) suffice to describe X_1 and X_2?



3. **Constrained signals**
What is the channel capacity of a binary symmetric channel with crossover probability $p = 3/4$ and "energy" constraint $EX^2 \leq 1/3$, $\mathcal{X} = \{0, 1\}$?

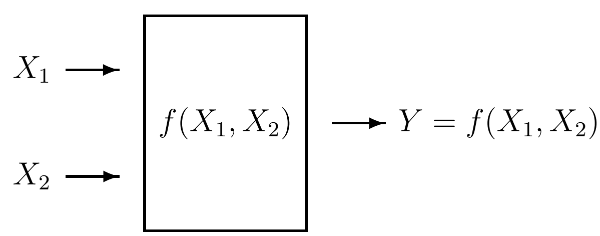
4. **Gaussian channel**



$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{n} X_i(W)^2 &\leq P \\ \frac{1}{n}\sum_{i=1}^{n} X_i(W) &\geq \alpha, \quad \text{for all } W \in 2^{nR}. \end{aligned}$$

What is the capacity of this additive Gaussian noise channel under the two constraints? What distribution on X achieves the capacity? The answer should be in closed form.

5. **Multiple access channel**



What is the capacity region of the deterministic multiple access channel if $f(X_1, X_2)$ is invertible, where $X_1 \in \{1, 2, \ldots, m_1\}$ and $X_2 \in \{1, 2, \ldots .m_2\}$?

6. **Velocity distribution**
Suppose we have atoms in a 10-dimensional box.

(a) What is the maximum entropy distribution on the velocity vector $\mathbf{v} = (v_1, v_2, \ldots, v_{10}) \in \mathbb{R}^{10}$ subject to the kinetic energy constraint

$$E\frac{1}{2}m||\mathbf{V}||^2 = E_o?$$

(b) Suppose there is also a potential energy contribution. An atom at height $x \geq 0$ and velocity $\mathbf{v} \in \mathbb{R}^{10}$ has total energy $\frac{1}{2}m||\mathbf{v}||^2 + mgx$. Find the maximum entropy density $f(x, \mathbf{v})$ subject to the constraints $x \geq 0$ and

$$E\left(\frac{1}{2}m||\mathbf{V}||^2 + mgX\right) = E_o.$$

7. **Rate distortion**
What is the rate distortion function $R(D)$ for a Bernoulli(p) source and distortion function

$$d(x, \hat{x}) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 5 & 2 \\ 1 & 2 & 5 \end{array}$$

8. **Likelihood ratio**
Let $X_1, X_2, \ldots$ be i.i.d.$\sim Q(x)$. Suppose one performs a likelihood ratio test of P_1 vs P_2. However, neither is true.

(a) Find

$$\Pr\left\{\sum_{i=1}^{n} \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha\right\}$$

if $X_1, X_2, \ldots$ are i.i.d.$\sim Q(x)$.

(b) Conditioned on

$$\sum_{i=1}^{n} \frac{1}{n} \ln \frac{P_1(X_i)}{P_2(X_i)} \geq \alpha,$$

what distribution P^* does the data "look like?" Distinguish the two cases for Q :

$$\sum_x Q(x) \ln \frac{P_1(x)}{P_2(x)} \geq \alpha$$

and

$$\sum_x Q(x) \ln \frac{P_1(x)}{P_2(x)} \leq \alpha.$$

9. **Horse race**
A horse race has win probabilities $(p_1, p_2, \ldots, p_m)$. The bettor places bets $(b_1, b_2, \ldots, b_m)$, where $b_i \geq 0$ and $\sum_{i=1}^{m} b_i = 1$. The odds-maker offers odds o_i for 1, $i = 1, 2, \ldots, m$, where the odds are consistent with some probability mass function $(r_1, r_2, \ldots, r_m)$, with $r_i \geq 0$ and $\sum_{i=1}^{m} r_i = 1$, in the sense that $o_i = 1/r_i$, $i = 1, 2, \ldots, m$. Let

$$W = \sum_{i=1}^{m} p_i \ln(b_i o_i)$$

be the growth rate of wealth. Let

$$v = \min_r \max_b W.$$

Find the minimax strategies b^* and r^*, and the value v of this game.