

Solutions to Homework Set #8

1. Investment growth rate.

Let

$$X = \begin{cases} (1, a), & \text{with probability } 1/2 \\ (1, 1/a), & \text{with probability } 1/2 \end{cases},$$

where $a > 1$. This vector \mathbf{X} represents a stock market vector of cash vs. a hot stock.
Let

$$W(\mathbf{b}, F) = E \log \mathbf{b}^t \mathbf{X}$$

and

$$W^* = \max_{\mathbf{b}} W(\mathbf{b}, F)$$

be the growth rate.

- (a) Find the log optimal portfolio \mathbf{b}^* .
- (b) Find the growth rate W^* .
- (c) Find the asymptotic behavior of

$$S_n = \prod_{i=1}^n \mathbf{b}^t \mathbf{X}_i$$

for all \mathbf{b} .

Solution: Investment growth rate.

- (a) Let the portfolio be $(1 - b_2, b_2)$. Then

$$W(\mathbf{b}, F) = \frac{1}{2} \ln(1 - b_2 + ab_2) + \frac{1}{2} \ln(1 - b_2 + \frac{b_2}{a}). \quad (1)$$

Differentiating to find the maximum, we have

$$\frac{dW}{db_2} = \frac{1}{2} \frac{a-1}{1-b_2+ab_2} - \frac{1-\frac{1}{a}}{1-b_2+\frac{b_2}{a}} = 0 \quad (2)$$

Solving this equation, we get $b_2^* = \frac{1}{2}$. Hence the log optimal portfolio \mathbf{b}^* is $(\frac{1}{2}, \frac{1}{2})$.

(b) The optimal growth rate $W^* = W(\mathbf{b}^*, F)$ is

$$W^* = \frac{1}{2} \ln \left(\frac{1}{2} + \frac{a}{2} \right) + \frac{1}{2} \ln \left(\frac{1}{2} + \frac{1}{2a} \right) \quad (3)$$

$$= \frac{1}{2} \ln \frac{(1+a)^2}{a} - \ln 2. \quad (4)$$

(c) The asymptotic behavior of an infinite product of i.i.d. terms is essentially determined by the expected log of the individual terms.

$$S_n = \prod_{i=1}^n \mathbf{b}^t \mathbf{X}_i \quad (5)$$

$$= e^{n \frac{1}{n} \sum_{i=1}^n \ln \mathbf{b}^t \mathbf{X}_i} \quad (6)$$

$$\rightarrow e^{n E \ln \mathbf{b}^t \mathbf{X}} \quad (7)$$

$$= e^{n W(\mathbf{b}, F)}, \quad (8)$$

where the convergence is with probability 1 by the strong law of large numbers. We can substitute for $W(\mathbf{b}, F)$ from (1).