Homework Set #6

1. **One bit quantization of a single Gaussian random variable**
Let $X \sim \mathcal{N}(0, \sigma^2)$ and let the distortion measure be squared error. Here we do not allow block descriptions. Show that the optimum reproduction points for 1 bit quantization are $\pm\sqrt{\frac{2}{\pi}}\sigma$, and that the expected distortion for 1 bit quantization is $\frac{\pi-2}{\pi}\sigma^2$.

   Compare this with the distortion rate bound $D = \sigma^2 2^{-2R}$ for $R = 1$.

2. **Rate distortion function with infinite distortion**
Find the rate distortion function $R(D) = \min I(X; \hat{X})$ for $X \sim$ Bernoulli $(\frac{1}{2})$ and distortion
$$d(x, \hat{x}) = \begin{cases} 0, & x = \hat{x}, \\ 1, & x = 1, \hat{x} = 0, \\ \infty, & x = 0, \hat{x} = 1, \end{cases}$$
where $x, \hat{x} \in \{0, 1\}$. Thus reconstructing a 0 by a 1 is never allowed.

3. **Properties of $R(D)$**
Here we wish to show that we can consider the minimum distortion to be zero. Consider a discrete source $X \in \mathcal{X} = \{1, 2, \ldots, m\}$ with distribution $p_1, p_2, \ldots, p_m$ and a distortion measure $d(x, \hat{x}), x, \hat{x} \in \{1, 2, \ldots, m\}$. Let $R(D)$ be the rate distortion function for this source and distortion measure. Let $d'(i, j) = d(i, j) - w_i$ be a new distortion measure and let $R'(D)$ be the corresponding rate distortion function. Show that $R'(D) = R(D + \bar{w})$, where $\bar{w} = \sum p_i w_i$, and use this to show that there is no essential loss of generality in assuming that $\min_{\hat{x}} d(i, \hat{x}) = 0$, i.e., for each $x \in \mathcal{X}$, there is one symbol $\hat{x}$ which reproduces the source with zero distortion.

4. **Erasure distortion**
Consider $X \sim$ Bernoulli$(\frac{1}{2})$, and let the distortion measure be given by the matrix
$$d(x, \hat{x}) = \begin{bmatrix} 0 & 1 & \infty \\ \infty & 1 & 0 \end{bmatrix},$$
where $x \in \{0, 1\}$ and $\hat{x} \in \{0, e, 1\}$. Calculate the rate distortion function for this source. Can you suggest a simple scheme to achieve any value of the rate distortion function for this source?

5. **Rate distortion with two constraints**
Let X_i be iid $\sim p(x)$. We are given two distortion functions $d_1(x, \hat{x})$ and $d_2(x, \hat{x})$. We wish to describe X^n at rate R and reconstruct it with distortions $Ed_1(X^n, \hat{X}_1^n) \leq D_1$, and $Ed_2(X^n, \hat{X}_2^n) \leq D_2$, as shown here:

$$X^n \longrightarrow i(X^n) \longrightarrow (\hat{X}_1^n(i), \hat{X}_2^n(i))$$

$$\begin{aligned} D_1 &= Ed_1(X_1^n, \hat{X}_1^n) \\ D_2 &= Ed_2(X_1^n, \hat{X}_2^n). \end{aligned}$$

Here $i(\cdot)$ takes on 2^{nR} values. What is the rate distortion function $R(D_1, D_2)$?

6. **Adding a column to the distortion matrix.**
Let $R(D)$ be the rate distortion function for an i.i.d. process with probability mass function $p(x)$ and distortion function $d(x, \hat{x}), x \in \mathcal{X}, \hat{x} \in \hat{\mathcal{X}}$. Now suppose that we add a new reproduction symbol $\hat{x}_0$ to $\hat{\mathcal{X}}$ with associated distortion $d(x, \hat{x}_0), x \in \mathcal{X}$. Does this increase or decrease $R(D)$ and why?

7. **Bounds on the rate distortion function for squared error distortion.**
For the case of a continuous random variable X with mean zero and variance σ^2 and squared error distortion, show that

$$h(X) - \frac{1}{2}\log(2\pi e)D \quad \leq \quad R(D) \quad \leq \quad \frac{1}{2}\log\frac{\sigma^2}{D}.$$

What does this imply about how easy or hard Gaussian random variables are to describe? Comment on *both* inequalities.