EE376B | Handout #11
Information Theory | Thursday, April 23, 2009
Prof. T. Cover | Due Tuesday, May 5, 2009

Homework Set #4

1. **The cooperative capacity of a multiple access channel.**

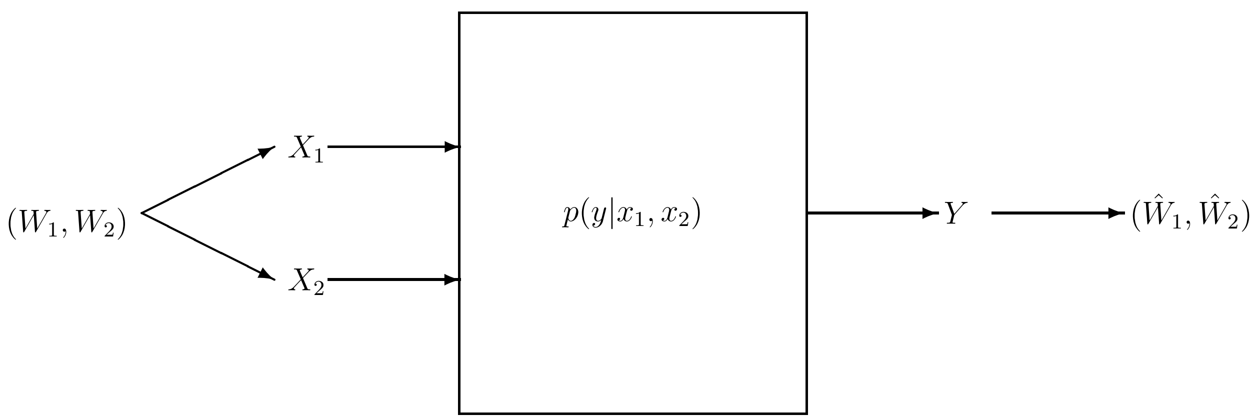


Figure 1: Multiple access channel with cooperating senders.

   (a) Suppose X_1 and X_2 have access to *both* indices $W_1 \in \{1, 2^{nR_1}\}, W_2 \in \{1, 2^{nR_2}\}$. Thus the codewords $\mathbf{X}_1(W_1, W_2), \mathbf{X}_2(W_1, W_2)$ depend on both indices. Find the capacity region.

   (b) Evaluate this region for the binary erasure multiple access channel $Y = X_1 + X_2$, $X_i \in \{0, 1\}$. Compare to the non-cooperative region.

2. **Square channel**.
   What is the capacity of the following multiple access channel?

$$\begin{aligned} X_1 &\in \{-1, 0, 1\} \\ X_2 &\in \{-1, 0, 1\} \\ Y &= X_1^2 + X_2^2 \end{aligned}$$

(a) Find the capacity region.

(b) Describe $p^*(x_1), p^*(x_2)$ achieving a point on the boundary of the capacity region.

(c) What is the capacity if $Y = X_1X_2$?

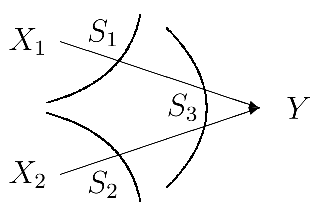
3. **Cut-set interpretation of capacity region of multiple access channel.** For the multiple access channel we know that (R_1, R_2) is achievable if

$$R_1 < I(X_1; Y \mid X_2), \tag{1}$$
$$R_2 < I(X_2; Y \mid X_1), \tag{2}$$
$$R_1 + R_2 < I(X_1, X_2; Y), \tag{3}$$

for X_1, X_2 independent. Show, for X_1, X_2 independent, that

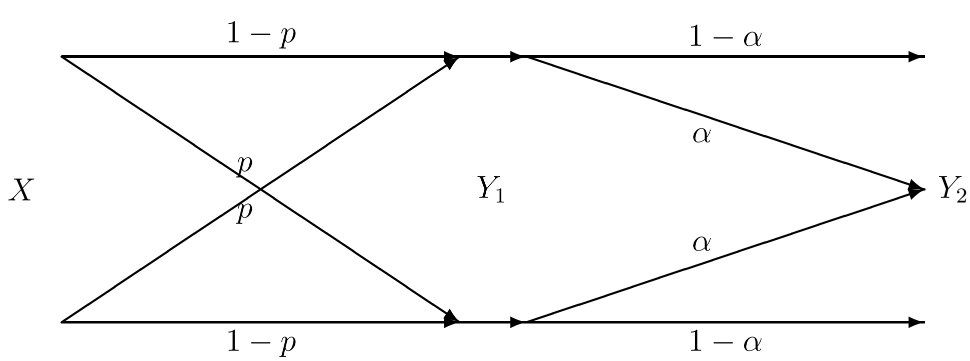
$$I(X_1; Y \mid X_2) = I(X_1; Y, X_2).$$

Thus R_1 is less than the mutual information between X_1 and everything else.



Interpret the information bounds as bounds on the rate of flow across cutsets S_1, S_2 and S_3.

4. **Degraded broadcast channel** Find the capacity region for the degraded broadcast channel below.



5. **Slepian-Wolf for deterministically related sources**
   Find and sketch the Slepian-Wolf rate region for the simultaneous data compression of (X, Y), where $y = f(x)$, and f is a given deterministic function.

6. **Converse for the degraded broadcast channel.**
   The following chain of inequalities proves the converse for the degraded discrete memoryless broadcast channel. We are given a degraded broadcast channel $p(x)p(y|x)p(z|y)$. Suppose we have a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes with (arithmetic average) probability of error tending to zero. Provide reasons for each of the labeled inequalities.

   Setup for converse for degraded broadcast channel capacity:

   Let W_1 and W_2 be independent and uniformly drawn over 2^{nR_1} and 2^{nR_2} respectively.

$$(W_1, W_2) \to X^n(W_1, W_2) \to Y^n \to Z^n$$

   Encoding $f_n : 2^{nR_1} \times 2^{nR_2} \to X^n$

   Decoding: $g_n : Y^n \to 2^{nR_1}, \qquad h_n : Z^n \to 2^{nR_2}$

   Let $U_i = (W_2, Y^{i-1})$. Then

$$\begin{aligned}
nR_2 \quad &\dot{\le}_{\text{Fano}} \quad I(W_2; Z^n) && (4)\\
&\overset{(a)}{=} \quad \sum_{i=1}^{n} I(W_2; Z_i \mid Z^{i-1}) && (5)\\
&\overset{(b)}{=} \quad \sum_i (H(Z_i \mid Z^{i-1}) - H(Z_i \mid W_2, Z^{i-1})) && (6)\\
&\overset{(c)}{\le} \quad \sum_i (H(Z_i) - H(Z_i \mid W_2, Z^{i-1}, Y^{i-1})) && (7)\\
&\overset{(d)}{\le} \quad \sum_i (H(Z_i) - H(Z_i \mid W_2, Y^{i-1})) && (8)\\
&\overset{(e)}{=} \quad \sum_{i=1}^{n} I(U_i; Z_i). && (9)
\end{aligned}$$

Continuation of converse. Give reasons for the labeled inequalities:

$$
\begin{aligned}
nR_1 \quad &\dot{\leq}_{\text{Fano}} \quad I(W_1; Y^n) && (10)\\
&\overset{(f)}{\leq} \quad I(W_1; Y^n, W_2) && (11)\\
&\overset{(g)}{\leq} \quad I(W_1; Y^n \mid W_2) && (12)\\
&\overset{(h)}{=} \quad \sum_{i-1}^{n} I(W_1; Y_i \mid Y^{i-1}, W_2) && (13)\\
&\overset{(i)}{\leq} \quad \sum_{i=1}^{n} I(X_i; Y_i \mid U_i). && (14)
\end{aligned}
$$

(15)

7. **Multiple access.**

   (a) Find the capacity region for the multiple access channel

   $$Y = X_1^{X_2}$$

   where

   $$X_1 \epsilon \{2, 4\} \ , \quad X_2 \epsilon \{1, 2\} \quad .$$

   (b) Suppose the range of X_1 is $\{1, 2\}$. Is the capacity region decreased?

8. **Broadcast channel.**

   (a) For the degraded broadcast channel $X \to Y_1 \to Y_2$, find the points a and b where the capacity region hits the R_1 and R_2 axes.

   (b) Show that $b \leq a$.