EE376B
Information Theory
Prof. T. Cover

Handout #7
Thursday, April 16, 2009
Due Thursday, April 23, 2009

Homework Set #3

1. **Multiple layer waterfilling**
   Let $C(x) = \frac{1}{2}\log(1+x)$ denote the channel capacity of a Gaussian channel with signal to noise ratio x. Show

$$C\left(\frac{P_1}{N}\right) + C\left(\frac{P_2}{P_1+N}\right) = C\left(\frac{P_1+P_2}{N}\right).$$

   This suggests that the first signal power P_1 acts as self noise for the second layer P_2.

2. **Parallel channels and waterfilling**
   Consider a pair of parallel Gaussian channels, i.e.,

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

   where

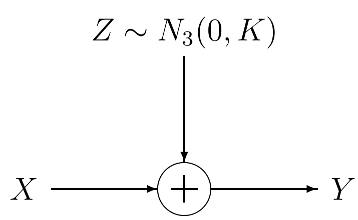
$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}\right),$$

   and there is a power constraint $E(X_1^2 + X_2^2) \leq P$. Assume that $\sigma_1^2 > \sigma_2^2$.

   (a) At what power does the channel stop behaving like a single channel with noise variance σ_2^2, and begin behaving like a pair of channels, ie., at what power does the worst channel become useful?

   (b) What is the capacity $C(P)$ for large P?

3. **Vector channel**
   Consider the 3 input 3 output Gaussian channel



where $X, Y, Z \in \mathbb{R}^3$, $E||X||^2 = E(X_1^2 + X_2^2 + X_3^2) \leq P$, and $Z \sim N_3(0, K)$. Find the capacity for

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}.$$

4. **Filtered noise**
   We consider the previous vector channel where Z is filtered white noise

$$Z = BU,$$

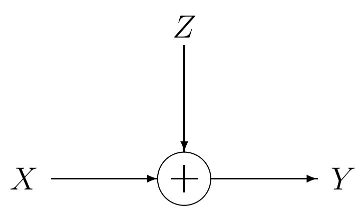
   where $U \sim N_3(0, I)$ is white Gaussian noise and

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 9 \end{bmatrix}.$$

   (a) Find the capacity.

   (b) How would you signal over this channel?

5. **A mutual information game**
   Consider the following channel:



   Throughout this problem we shall constrain the signal power

$$EX = 0, \qquad EX^2 = P,$$

   and the noise power

$$EZ = 0, \qquad EZ^2 = N,$$

   and assume that X and Z are independent. The channel capacity is given by $I(X; X + Z)$.

   Now for the game. The noise player chooses a distribution on Z to minimize $I(X; X + Z)$, while the signal player chooses a distribution on X to maximize $I(X; X + Z)$.

Letting $X^* \sim \mathcal{N}(0,P), Z^* \sim \mathcal{N}(0,N)$, show that Gaussian X^* and Z^* satisfy the saddlepoint conditions

$$I(X;X+Z^*) \le I(X^*;X^*+Z^*) \le I(X^*;X^*+Z).$$

Thus

$$\min_Z \max_X I(X;X+Z) = \max_X \min_Z I(X;X+Z) = \frac{1}{2}\log\left(1+\frac{P}{N}\right),$$

and the game has a value. In particular, a deviation from normal for either player worsens the mutual information from that player's standpoint. Can you discuss the implications of this?

Note: Part of the proof hinges on the entropy power inequality from Chapter 16, which states that if $\mathbf{X}$ and $\mathbf{Y}$ are independent random n-vectors with densities, then

$$e^{\frac{2}{n}h(\mathbf{X}+\mathbf{Y})} \ge e^{\frac{2}{n}h(\mathbf{X})} + e^{\frac{2}{n}h(\mathbf{Y})}.$$

6. **Additive noise channel**
   This problem has an instructive answer. Consider the channel $Y = X + Z$, where X is the transmitted signal with power constraint P, Z is independent additive noise, and Y is the received signal. Let

$$Z = \begin{cases} 0, & \text{with prob. } 1/10 \\ Z^*, & \text{with prob. } 9/10 \end{cases},$$

   where $Z^* \sim N(0,N)$. Thus Z has a mixture distribution which is the mixture of a Gaussian distribution and a degenerate distribution with mass 1 at 0.

   (a) What is the capacity of this channel?

   (b) How would you signal in such a manner as to achieve capacity?

7. **Estimation.**
   Here is the estimation counterpart to Fano's inequality. Let X be a random variable with differential entropy $h(X)$. Let $\hat{X}$ be an estimate of X, and let $E(X-\hat{X})^2$ be the expected prediction error.

   Given side information Y and estimator $\hat{X}(Y)$, show

$$E(X-\hat{X}(Y))^2 \ge \frac{1}{2\pi e}e^{2h(X|Y)}.$$

   What are conditions for equality?