Homework Set #2

1. **Maximum entropy processes.**
   Find the maximum entropy rate stochastic processes $\{X_i\}_{-\infty}^{\infty}$ subject to the constraints:

   (a) $EX_i^2 = 1, \quad i = 1, 2, \ldots,$

   (b) $EX_i^2 = 1,\ EX_iX_{i+1} = \frac{1}{2}, \quad i = 1, 2, \ldots.$

   Find the maximum entropy spectrum for the processes in parts (a) and (b).

2. **Maximum entropy discrete processes.**

   (a) Find the maximum entropy rate binary stochastic process $\{X_i\}_{i=-\infty}^{\infty},\ \ X_i \in \{0, 1\}$, satisfying $\Pr\{X_i = X_{i+1}\} = \frac{1}{3}$, for all i.

   (b) What is the resulting entropy rate?

3. **Processes With Fixed Marginals**
   Consider the set of all densities with fixed pairwise marginals

$$f_{12}(x_1, x_2), f_{23}(x_2, x_3), \ldots, f_{n-1,n}(x_{n-1}, x_n).$$

   Show that the maximum entropy process with these marginals is the first-order (possibly time-varying) Markov process with these marginals. Identify the maximizing $f^*(x_1, x_2, \ldots, x_n)$.

4. **Maximum entropy of sums.**
   Let $Y = X_1 + X_2$. Find the maximum entropy (over all distributions on X_1 and X_2) of Y under the constraint $EX_1^2 = P_1, \quad EX_2^2 = P_2$,

   (a) if X_1 and X_2 are independent.

   (b) if X_1 and X_2 are allowed to be dependent.

5. **Hadamard.**
Let K be a $2n \times 2n$ nonnegative definite symmetric matrix. Show

$$\det(K) \le \prod_{i=1}^{n} \det\left(K(2i-1, 2i)\right),$$

where $K(i,j)$ denotes the 2×2 submatrix

$$\begin{pmatrix} K_{ii} & K_{ij} \\ K_{ji} & K_{jj} \end{pmatrix}.$$

6. **Maximum entropy.**

   (a) What is the parametric form maximum entropy density $f(x)$ satisfying the two conditions

   $$EX^8 = a$$
   $$EX^{16} = b.$$

   Don't solve for the λ's.

   (b) What is the maximum entropy density satisfying the condition

   $$E(X^8 + X^{16}) = a + b \quad ?$$

   Again, don't solve for the λ's.

7. **Mutual information.**
Recall that

$$I(X;Y) = \sup_{[\ ],[\ ]} I([X];[Y])$$

to calculate $I(X;Y)$, where the distribution of (X,Y) is the $(\lambda, 1-\lambda)$ mixture of the density $f(x,y)$ and the probability mass function

$$(X,Y) = \begin{cases} (1,1), & \frac{1}{2}q \\ (1,2), & \frac{1}{2}p \\ (2,1), & \frac{1}{2}p \\ (2,2), & \frac{1}{2}q \end{cases}.$$