Homework Set #1

1. **Differential entropy**
   Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

   (a) The Laplace density, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$. Relate this to the entropy of the exponential density $\lambda e^{-\lambda x}$, $x \geq 0$.

   (b) The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with means μ_i and variances σ_i^2, $i = 1, 2$.

2. **Maximum entropy with marginals**
   What is the maximum entropy probability mass function $p_{i,j}(x, y)$ with the following marginals? You may wish to guess and verify a more general result.

	y_1	y_2	y_3	
x_1	p_{11}	p_{12}	p_{13}	1/2
x_2	p_{21}	p_{22}	p_{23}	1/4
x_3	p_{31}	p_{32}	p_{33}	1/4
	2/3	1/6	1/6	

3. **Maximum entropy of atmosphere**
   Maximize $h(Z, V_x, V_y, V_z)$, $Z \geq 0$, $(V_x, V_y, V_z) \in R^3$, subject to the energy constraint $E(\frac{1}{2}m \parallel V \parallel^2 + mgZ) = E_0$. Show that the resulting distribution yields

$$E\frac{1}{2}m \parallel V \parallel^2 = \frac{3}{5}E_0$$

$$EmgZ = \frac{2}{5}E_0.$$

   Thus $\frac{2}{5}$ of the energy is stored in the potential field, regardless of its strength g.

4. **Gaussian mutual information**
Suppose that (X, Y, Z) are jointly Gaussian and that $X \to Y \to Z$ forms a Markov chain. Let X and Y have correlation coefficient ρ_1 and let Y and Z have correlation coefficient ρ_2. Find $I(X; Z)$.

5. **Maximum entropy**
Find the maximum entropy density f satisfying $EX = \alpha_1, E \ln X = \alpha_2$. That is,

$$\text{maximize} \quad h(f)$$

subject to $\int x f(x)\, dx = \alpha_1, \int (\ln x) f(x)\, dx = \alpha_2$. What family of densities is this?

6. **Minimum relative entropy $D(P \parallel Q)$ under constraints on P.**
We wish to find the (parametric form) of the probability mass function $P(x), \quad x \in \{1, 2, \ldots\}$, that minimizes the relative entropy $D(P \parallel Q)$ over all P such that $\sum P(x) g_i(x) = \alpha_i, \quad i = 1, 2, \ldots$.

   (a) Use Lagrange multipliers to guess that

   $$P^*(x) = Q(x) e^{\sum_{i=1}^{\infty} \lambda_i g_i(x) + \lambda_0}$$

   achieves this minimum if there exist λ_i's satisfying the α_i constraints. This generalizes the theorem on maximum entropy distributions subject to constraints.

   (b) Verify that P^* minimizes $D(P \parallel Q)$.

7. **Every density is a maximum entropy density for some constraint**
We wish to show that any density f_0 can be considered to be a maximum entropy density. Let $f_0(x)$ be a density and consider the problem of maximizing $h(f)$ subject to the constraint

$$\int f(x) r(x)\, dx = \alpha$$

where $r(x) = \ln f_0(x)$. Show that there is a choice of α, $\alpha = \alpha_0$, such that the maximizing distribution is $f^*(x) = f_0(x)$. Thus $f_0(x)$ is indeed a maximum entropy density under the constraint $\int f \ln f_0 = \alpha_0$.