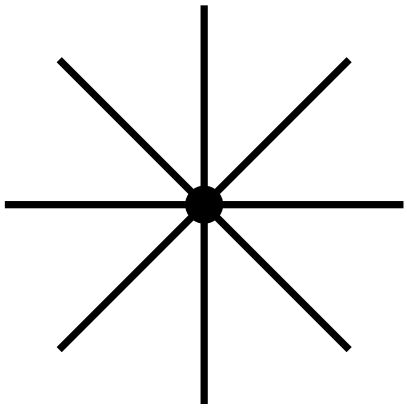
EE 376A/Stat 376A
Information Theory
Prof. T. Cover

Handout #31
Thursday, March 12, 2009

Practice Final Examination

1. **Graph.**
   What is the entropy rate of a random walk on the star graph with a central hub node and n edges:



2. **Optimal code when entropy is infinite.**
   Let X be integer valued with $H(X) = -\sum_{i=1}^{\infty} p_i \log p_i = \infty$. Thus the expected (binary) description length $L = \sum p_i l_i$ is infinite, even for the Shannon ideal codeword length $l_i^* = \log \frac{1}{p_i}$.

   Show nonetheless that $\{l_i^*\}$ is better than $\{l_i\}$ for any other instantaneous code in the sense that $\sum_{i=1}^{\infty} p_i(l_i - l_i^*) \geq 0$, for all $\{l_i\}$ satisfying the Kraft inequality.

3. **Huffman code.**

   (a) Find the binary Huffman code for $p = \{\frac{7}{20}, \frac{4}{20}, \frac{4}{20}, \frac{3}{20}, \frac{2}{20}\}$.

   (b) *Guess* the optimal (minimal expected description length) binary code for the integer valued random variable X, where $\Pr\{X = i\} = p^i q, \quad i = 0, 1, \ldots,$ and $p = .4$.

4. **Random walk.**
   Let X_i be i.i.d. $\sim X$, where

$$X = \begin{cases} 1 & , \quad p \\ -1 & , \quad 1-p \end{cases}$$

   Let $S_n = \sum_{i=1}^{n} X_i$ be the position of the random walk at time n

   (a) Is $\{S_n\}_{n=1}^{\infty}$ stationary? Yes or no.

   (b) Does $\{S_n\}$ have an entropy rate? If so, what is it?

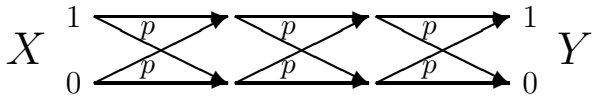
5. **Mutual information.**
   Consider all probability mass functions on (U, X, Y) of the form $p(u, x, y) = p(u, x)p(y|x)$.

   (a) Is $I(U;Y) \geq, =,$ or $\leq I(X;Y)$?

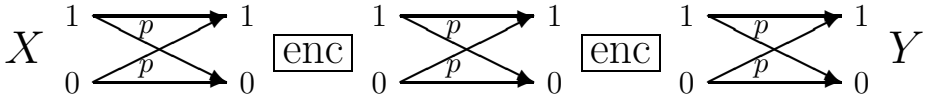
   (b) Prove it.

6. **Capacity.**
Find the capacities of the following channels.

(a) $Y = X \oplus Z_1 \oplus Z_2 \oplus Z_3$, where $X \in \{0, 1\}$ and Z_1, Z_2, Z_3 are independent Bernoulli(p).

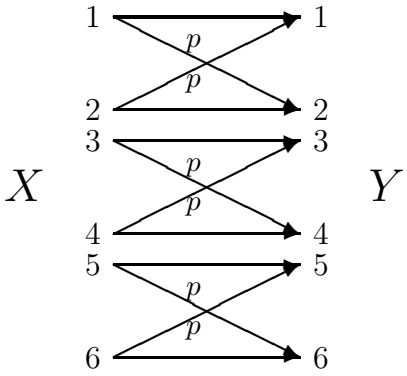


(b) Cascade of 3 BSC(p)'s with encoding and decoding between each.

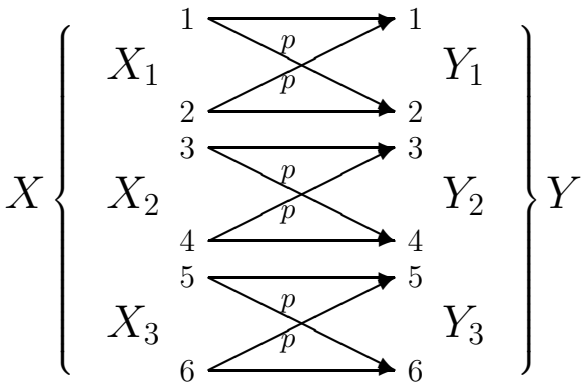


(c) $Y = (X \oplus Z_1, X \oplus Z_2, X \oplus Z_3)$, where $Z_k \sim$ Bern(p). Thus Y is 3 independent looks at X.

(d) Parallel BSC's, send a symbol on only one.

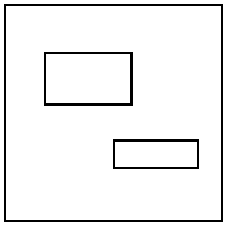


(e) Parallel BSC's, send a symbol on each.

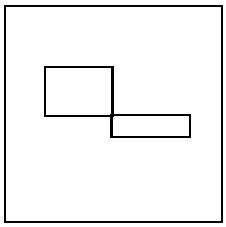


7. **Kolmogorov complexity.**
Images are displayed on an $n \times n$ screen. What is the Kolmogorov complexity $K(x|n)$ of the following?

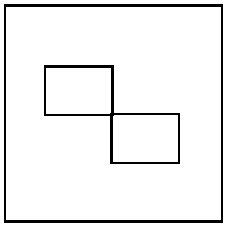
(a) arbitrary rectangles

$x =$ 

(b) arbitrary rectangles meeting at center

$x =$ 

(c) congruent rectangles meeting at center

$x =$ 

8. **Random "20" questions.**
Let X be uniformly distributed over $\{1, 2, \ldots, m\}$. Assume $m = 2^n$. We ask random questions: Is $X \in S_1$? Is $X \in S_2$?...until only one integer remains. All 2^m subsets of $\{1, 2, \ldots, m\}$ are equally likely.

(a) How many deterministic questions are needed to determine X?

(b) Without loss of generality, suppose that $X = 1$ is the object. Ask the random questions $S_1, S_2, \ldots$. What is the probability that object 2 yields the same answers for k random questions as object 1?

(c) What is the expected number of objects in $\{2, 3, \ldots, m\}$ that have the same answers to the k random questions as does the correct object 1?

(d) Now let $k = n + \sqrt{n}$. Use Markov's inequality $\Pr\{X \geq t\mu\} \leq \frac{1}{t}$, to show that the probability of error (one or more wrong objects remaining) goes to zero as $n \longrightarrow \infty$.