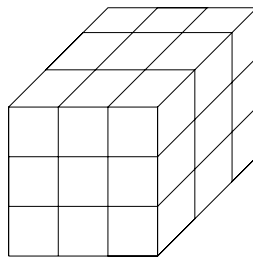


**Solutions to Sample Midterm Examination (Originally given in 2004)**

1. (20 points) **3-D Maze.**

A bird is lost in a  $3 \times 3 \times 3$  cubical maze. The bird flies from room to room going to adjoining rooms with equal probability. To be specific, the corner rooms have 3 exits.

What is the entropy rate of this random walk?



**Solution: 3D Maze.**

The entropy rate of a random walk on a graph with equal weights is given by equation 4.41 in the text:

$$H(\mathcal{X}) = \log(2E) - H\left(\frac{E_1}{2E}, \dots, \frac{E_m}{2E}\right)$$

There are 8 corners, 12 edges, 6 faces and 1 center. Corners have 3 edges, edges have 4 edges, faces have 5 edges and centers have 6 edges. Therefore, the total number of edges  $E = 54$ . So,

$$\begin{aligned} H(\mathcal{X}) &= \log(108) + 8 \left( \frac{3}{108} \log \frac{3}{108} \right) + 12 \left( \frac{4}{108} \log \frac{4}{108} \right) + 6 \left( \frac{5}{108} \log \frac{5}{108} \right) + 1 \left( \frac{6}{108} \log \frac{6}{108} \right) \\ &= 2.03 \text{ bits} \end{aligned}$$

2. (20 points) **Limits.**

Find the limit of

$$p^{1/n}(X_1, \dots, X_n)$$

where the  $X_i$  are *i.i.d.* with geometric distribution

$$\Pr\{X = i\} = \left(\frac{1}{2}\right)^i \quad i = 1, 2, 3, \dots$$

**Solution: Limits.**

Begin by observing that independence gives us

$$p^{1/n}(X_1, \dots, X_n) = \left(\prod_{i=1}^n p(X_i)\right)^{1/n}.$$

If we now take the log

$$\begin{aligned} \lim_{n \rightarrow \infty} \log p^{1/n}(X_1, \dots, X_n) &= \lim_{n \rightarrow \infty} \log \left(\prod_{i=1}^n p(X_i)\right)^{1/n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log p(X_i) \\ &\stackrel{(a)}{=} \mathbf{E}(\log p(X)) \quad \text{w.p. 1} \end{aligned}$$

where (a) comes from the strong law of large number. Hence

$$\lim_{n \rightarrow \infty} \log p^{1/n}(X_1, \dots, X_n) = -H(X) \quad \text{w.p. 1.}$$

Therefore,

$$\lim_{n \rightarrow \infty} p^{1/n}(X_1, \dots, X_n) = 2^{-H(X)} \quad \text{w.p. 1.}$$

Here

$$\begin{aligned} H(X) &= -\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \log \left(\frac{1}{2}\right)^i \\ &= -\sum_{i=1}^{\infty} -i \left(\frac{1}{2}\right)^i \\ &= \sum_{i=1}^{\infty} i \left(\frac{1}{2}\right)^i \\ &= \mathbf{E}(\mathbb{X}) \\ &= 2 \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} p^{1/n}(X_1, \dots, X_n) = \frac{1}{4} \quad \text{w.p. 1.}$$

3. (20 points) **Huffman code.**

Given the probabilities  $\mathbf{p} = (\frac{10}{45}, \frac{8}{45}, \frac{7}{45}, \frac{6}{45}, \frac{5}{45}, \frac{5}{45}, \frac{3}{45}, \frac{1}{45})$ , find the Huffman code for

- (a) (10 points) a binary Huffman code.
- (b) (10 points) a fourary Huffman code.

**Solution: Huffman code.**

- (a) For the binary Huffman we have

| Codeword |       |    |    |    |    |    |    |    |    |
|----------|-------|----|----|----|----|----|----|----|----|
| 01       | $x_1$ | 10 | 10 | 10 | 11 | 15 | 19 | 26 | 45 |
| 111      | $x_2$ | 8  | 8  | 9  | 10 | 11 | 15 | 19 |    |
| 110      | $x_3$ | 7  | 7  | 8  | 9  | 10 | 11 |    |    |
| 101      | $x_4$ | 6  | 6  | 7  | 8  | 9  |    |    |    |
| 100      | $x_5$ | 5  | 5  | 6  | 7  |    |    |    |    |
| 001      | $x_6$ | 5  | 5  | 5  |    |    |    |    |    |
| 0001     | $x_7$ | 3  | 4  |    |    |    |    |    |    |
| 0000     | $x_8$ | 1  |    |    |    |    |    |    |    |

- (b) For the fourary we have

|     |       |    |    |   |    |    |
|-----|-------|----|----|---|----|----|
| a   | $x_1$ | 10 | 10 | 0 | 10 | 45 |
| b   | $x_2$ | 8  | 8  |   | 8  |    |
| c   | $x_3$ | 7  | 7  |   | 7  |    |
| da  | $x_4$ | 6  | 6  |   | 20 |    |
| db  | $x_5$ | 5  | 5  |   |    |    |
| dc  | $x_6$ | 5  | 5  |   |    |    |
| dda | $x_7$ | 3  | 4  |   |    |    |
| ddb | $x_8$ | 1  |    |   |    |    |
|     | $d_1$ | 0  |    |   |    |    |
|     | $d_2$ | 0  |    |   |    |    |

4. (20 points) **Optimal Huffman.**

The following Huffman code achieves the entropy bound  $L = H$  for what probability mass function?

$$\{0, 100, 101, 110, 1110, 1111\}$$

**Solution: Optimal Huffman.**

We know that the for the given binary Huffman code to be optimal  $l(x) = -\log p(x)$ ,

where  $l(x)$  is the length of the codeword for  $x$ . Hence we have the following optimal distribution

| Codeword | Probability |
|----------|-------------|
| 0        | 1/2         |
| 100      | 1/8         |
| 101      | 1/8         |
| 110      | 1/8         |
| 1110     | 1/16        |
| 1111     | 1/16        |

5. (20 points) **Time symmetry.**

Let  $\{X_n\}$  be a stationary Markov process. We condition on  $(X_0, X_1)$  and look into the past and future. For what index  $k$  is

$$H(X_{-n}|X_0, X_1) = H(X_k|X_0, X_1)?$$

Give the argument.

**Solution: Time symmetry.**

The trivial solution is  $k = -n$ . To find other possible values of  $k$  we expand

$$\begin{aligned}
 H(X_{-n}|X_0, X_1) &= H(X_{-n}, X_0, X_1) - H(X_0, X_1) \\
 &= H(X_{-n}) + H(X_0, X_1|X_{-n}) - H(X_0, X_1) \\
 &= H(X_{-n}) + H(X_0|X_{-n}) + H(X_1|X_0, X_{-n}) - H(X_0, X_1) \\
 &\stackrel{(a)}{=} H(X_{-n}) + H(X_0|X_{-n}) + H(X_1|X_0) - H(X_0, X_1) \\
 &= H(X_{-n}) + H(X_0|X_{-n}) - H(X_0) \\
 &\stackrel{(b)}{=} H(X_0) + H(X_0|X_{-n}) - H(X_0) \\
 &\stackrel{(c)}{=} H(X_n|X_0) \\
 &\stackrel{(d)}{=} H(X_n|X_0, X_{-1}) \\
 &\stackrel{(e)}{=} H(X_{n+1}|X_1, X_0)
 \end{aligned}$$

where (a) and (d) come from Markovity and (b), (c) and (e) come from stationarity. Hence  $k = n + 1$  is also a solution. There are no other solution since for any other  $k$ , we can construct a periodic Markov process as a counter example. Therefore  $k \in \{-n, n + 1\}$ .

6. (20 points) **Fano.**

We are given the following joint distribution on  $(X, Y)$

|   | Y              |                |                |
|---|----------------|----------------|----------------|
| X | a              | b              | c              |
| 1 | $\frac{1}{6}$  | $\frac{1}{12}$ | $\frac{1}{12}$ |
| 2 | $\frac{1}{12}$ | $\frac{1}{6}$  | $\frac{1}{12}$ |
| 3 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{6}$  |

Let  $\hat{X}(Y)$  be an estimator for  $X$  (based on  $Y$ ) and let  $P_e = \Pr\{\hat{X}(Y) \neq X\}$ .

- (a) (10 points) Find the minimum probability of error estimator  $\hat{X}(Y)$  and the associated  $P_e$ .
- (b) (10 points) Evaluate Fano's inequality for this problem and compare.

**Solution: Fano.**

- (a) From inspection we see that

$$\hat{X}(y) = \begin{cases} 1 & y = a \\ 2 & y = b \\ 3 & y = c \end{cases}$$

Hence the associated  $P_e$  is the sum of  $P(1, b)$ ,  $P(1, c)$ ,  $P(2, a)$ ,  $P(2, c)$ ,  $P(3, a)$  and  $P(3, b)$ . Therefore,  $P_e = 1/2$ .

- (b) From Fano's inequality we know

$$P_e \geq \frac{H(X|Y) - 1}{\log |\mathcal{X}|}.$$

Here,

$$\begin{aligned} H(X|Y) &= H(X|Y = a) \Pr\{y = a\} + H(X|Y = b) \Pr\{y = b\} + H(X|Y = c) \Pr\{y = c\} \\ &= H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \Pr\{y = a\} + H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \Pr\{y = b\} + H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \Pr\{y = c\} \\ &= H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) (\Pr\{y = a\} + \Pr\{y = b\} + \Pr\{y = c\}) \\ &= H\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \\ &= 1.5 \text{ bits.} \end{aligned}$$

Hence

$$P_e \geq \frac{1.5 - 1}{\log 3} = .316.$$

Hence our estimator  $\hat{X}(Y)$  is not very close to Fano's bound in this form. If  $\hat{X} \in \mathcal{X}$ , as it does here, we can use the stronger form of Fano's inequality to get

$$P_e \geq \frac{H(X|Y) - 1}{\log(|\mathcal{X}|-1)}.$$

and

$$P_e \geq \frac{1.5 - 1}{\log 2} = \frac{1}{2}.$$

Therefore our estimator  $\hat{X}(Y)$  is actually quite good.

7. (20 points) **Dutch book.**

Consider a horse race with  $m = 2$  horses,

$$\begin{aligned} X &= 1, 2 \\ P &= 1/2, 1/2 \\ \text{Odds (for one)} &= 10, 30 \\ \text{Bets} &= b, 1 - b. \end{aligned}$$

The odds are super fair.

- (a) (10 points) There is a bet  $b$  which guarantees the same payoff regardless of which horse wins. Such a bet is called a Dutch book. Find this  $b$  and the associated wealth factor  $S(X)$ .
- (b) (10 points) What is the maximum growth rate of the wealth for this gamble? Compare it to the growth rate for the Dutch book.

**Solution: Dutch book.**

(a)

$$\begin{aligned} 10b_D + 30(1 - b_D) &= 40b_D \\ 40b_D &= 30 \\ b_D &= 3/4. \end{aligned}$$

Therefore,

$$\begin{aligned} W(b_D, P) &= \frac{1}{2} \log \left( 10 \frac{3}{4} \right) + \frac{1}{2} \log \left( 30 \frac{1}{4} \right) \\ &= 2.91 \end{aligned}$$

and

$$S_D(X) = 2^{W(b_D, P)} = 7.5.$$

(b) In general,

$$W(b, p) = \frac{1}{2} \log(10b) + \frac{1}{2} \log(30(1 - b)).$$

Setting the  $\frac{\partial W}{\partial b}$  to zero we get

$$\begin{aligned} \frac{1}{2} \left( \frac{10}{10b^*} \right) + \frac{1}{2} \left( \frac{-30}{30 - 30b^*} \right) &= 0 \\ \frac{1}{2b^*} + \frac{1}{2(b^* - 1)} &= 0 \\ \frac{(b^* - 1) + b^*}{2b^*(b^* - 1)} &= 0 \\ \frac{2b^* - 1}{4b^*(1 - b^*)} &= 0 \\ b^* &= \frac{1}{2}. \end{aligned}$$

Hence

$$\begin{aligned} W^*(p) &= \frac{1}{2} \log(5) + \frac{1}{2} \log(15) = 3.11 \\ W(b_D, p) &= 2.91 \end{aligned}$$

and

$$\begin{aligned} S^* &= 2^{W^*} = 8.66 \\ S_D &= 2^{W_D} = 7.5 \end{aligned}$$