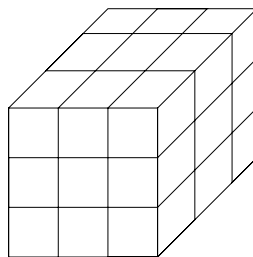


Sample Midterm Examination (Originally given in 2004)

1. (20 points) **3-D Maze.**

A bird is lost in a $3 \times 3 \times 3$ cubical maze. The bird flies from room to room going to adjoining rooms with equal probability. To be specific, the corner rooms have 3 exits.

What is the entropy rate of this random walk?



2. (20 points) **Limits.**

Find the limit of

$$p^{1/n}(X_1, \dots, X_n)$$

where the X_i are *i.i.d.* with geometric distribution

$$\Pr\{X = i\} = \left(\frac{1}{2}\right)^i \quad i = 1, 2, 3, \dots$$

3. (20 points) **Huffman code.**

Given the probabilities $\mathbf{p} = (\frac{10}{45}, \frac{8}{45}, \frac{7}{45}, \frac{6}{45}, \frac{5}{45}, \frac{5}{45}, \frac{3}{45}, \frac{1}{45})$, find the Huffman code for

(a) (10 points) a binary Huffman code.

(b) (10 points) a fourary Huffman code.

4. (20 points) **Optimal Huffman.**

The following Huffman code achieves the entropy bound $L = H$ for what probability mass function?

$$\{0, 100, 101, 110, 1110, 1111\}$$

5. (20 points) **Time symmetry.**

Let $\{X_n\}$ be a stationary Markov process. We condition on (X_0, X_1) and look into the past and future. For what index k is

$$H(X_{-n}|X_0, X_1) = H(X_k|X_0, X_1)?$$

Give the argument.

6. (20 points) **Fano.**

We are given the following joint distribution on (X, Y)

	Y			
X		a	b	c
1		$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{12}$
2		$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
3		$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$

Let $\hat{X}(Y)$ be an estimator for X (based on Y) and let $P_e = \Pr\{\hat{X}(Y) \neq X\}$.

- (a) (10 points) Find the minimum probability of error estimator $\hat{X}(Y)$ and the associated P_e .
- (b) (10 points) Evaluate Fano's inequality for this problem and compare.

7. (20 points) **Dutch book.**

Consider a horse race with $m = 2$ horses,

$$\begin{aligned} X &= 1, 2 \\ P &= 1/2, 1/2 \\ \text{Odds (for one)} &= 10, 30 \\ \text{Bets} &= b, 1 - b. \end{aligned}$$

The odds are super fair.

- (a) (10 points) There is a bet b which guarantees the same payoff regardless of which horse wins. Such a bet is called a Dutch book. Find this b and the associated wealth factor $S(X)$.
- (b) (10 points) What is the maximum growth rate of the wealth for this gamble? Compare it to the growth rate for the Dutch book.