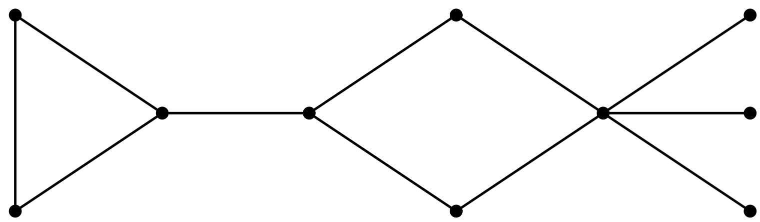
Sample Midterm Examination (Originally given in 2001)

1. (*20 points*) **Random walk on graph.**
   What is the entropy rate of a random walk on this undirected graph?



2. (*30 points*) **Huffman code.**
   Find the D-ary Huffman code for $\mathbf{p} = (\frac{8}{26}, \frac{8}{26}, \frac{7}{26}, \frac{1}{26}, \frac{1}{26}, \frac{1}{26})$.

   (a) (*10 points*) For $D = 2$.

   (b) (*10 points*) For $D = 4$.

   (c) (*10 points*) What is the entropy of the first question implied by the binary code in part (a)?

3. (*30 points*) **Fair to bent.**

   (a) (*10 points*) Describe how to use fair coin flips $Z_1, Z_2, Z_3, \ldots$ to generate a random variable

$$X = \begin{cases} 1 & \text{with probability } 1/4, \\ 2 & \text{with probability } 1/4, \\ 3 & \text{with probability } 1/4, \\ 4 & \text{with probability } 1/8, \\ 5 & \text{with probability } 1/16, \\ 6 & \text{with probability } 1/16. \end{cases}$$

   (b) (*10 points*) What is the expected number of flips required?

   (c) (*10 points*) What is $H(X)$?

4. (*20 points*) **Betting.**
Consider a horse race with win probabilities $(p_1, p_2, \ldots, p_m)$ and m-for-1 odds. Suppose the amount b_1 bet on the first horse is fixed. Find the bets $b_2, \ldots, b_m$ that maximize the growth rate of wealth.

5. (*60 points*) **Markov chain.**

$$P = [P_{ij}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Let X_1 be uniformly distributed over the states $\{0, 1, 2\}$. Let $\{X_i\}_1^\infty$ be a Markov chain with transition matrix P, thus $P(X_{n+1} = j | X_n = i) = P_{ij}, i, j \in \{0, 1, 2\}$.

(a) (*10 points*) Is $\{X_n\}$ stationary?

(b) (*10 points*) Find $\lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n)$.

Now consider the derived process $Z_1, Z_2, \ldots, Z_n$, where

$$\begin{aligned} Z_1 &= X_1 \\ Z_i &= X_i - X_{i-1} \pmod 3, \qquad i = 2, \ldots, n. \end{aligned}$$

Thus Z^n encodes the transitions, not the states.

(c) (*10 points*) Find $H(Z_1, Z_2, ..., Z_n)$.

(d) (*10 points*) Find $H(Z_n)$ and $H(X_n)$, for $n \geq 2$.

(e) (*10 points*) Find $H(Z_n | Z_{n-1})$ for $n \geq 2$.

(f) (*10 points*) Are Z_{n-1} and Z_n independent for $n \geq 2$?

6. (*30 points*) **AEP.**
Let $X_1, X_2, \ldots$ be an i.i.d. sequence of discrete random variables with entropy $H(X)$. Let

$$C_n(t) = \{x^n \in \mathcal{X}^n : p(x^n) \geq 2^{-nt}\}$$

denote the subset of n-sequences with probabilities $\geq 2^{-nt}$.

(a) (*15 points*) Show $|C_n(t)| \leq 2^{nt}$.

(b) (*15 points*) For what values of t does $P(\{X^n \in C_n(t)\}) \to 1$?