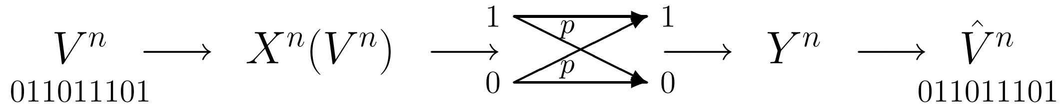
EE 376A
Information Theory
Prof. T. Cover

Handout #27
Thursday, February 26, 2009
Due *Tuesday,* March 10, 2009

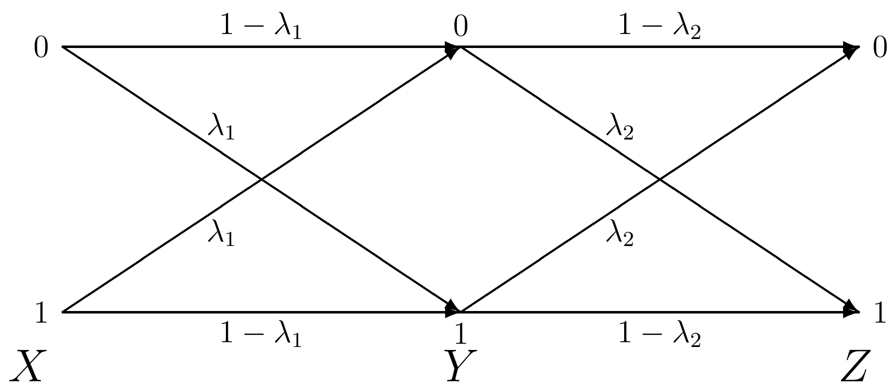
Homework Set #8

1. **Source and channel.**
We wish to encode a Bernoulli(α) process $V_1, V_2, \ldots$ for transmission over a binary symmetric channel with error probability p.



Find conditions on α and p so that the probability of error $P(\hat{V}^n \neq V^n)$ can be made to go to zero as $n \longrightarrow \infty$.

2. **Cascaded BSCs.**
Consider the two discrete memoryless channels $(\mathcal{X}, p_1(y|x), \mathcal{Y})$ and $(\mathcal{Y}, p_2(z|y), \mathcal{Z})$.
Let $p_1(y|x)$ and $p_2(z|y)$ be binary symmetric channels with crossover probabilities λ_1 and λ_2 respectively.



(a) What is the capacity C_1 of $p_1(y|x)$?

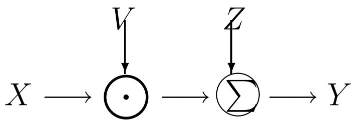
(b) What is the capacity C_2 of $p_2(z|y)$?

(c) We now cascade these channels. Thus $p_3(z|x) = \sum_y p_1(y|x)p_2(z|y)$. What is the capacity C_3 of $p_3(z|x)$? Show $C_3 \leq \min\{C_1, C_2\}$.

(d) Now let us actively intervene between channels 1 and 2, rather than passively transmitting y^n. What is the capacity of channel 1 followed by channel 2 if you are allowed to decode the output y^n of channel 1 and then reencode it as $\tilde{y}^n$ for transmission over channel 2? (Think $W \longrightarrow x^n(W) \longrightarrow Y^n \longrightarrow \tilde{y}^n(Y^n) \longrightarrow Z^n \longrightarrow \hat{W}$.)

(e) What is the capacity of the cascade in part c) if the receiver can view *both* Y and Z?

3. **Fading channel.**
Consider an additive noise fading channel



$$Y = XV + Z,$$

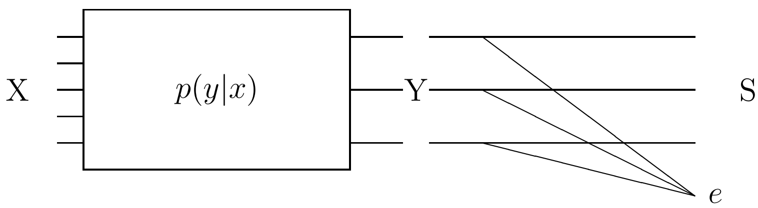
where Z is additive noise, V is a random variable representing fading, and Z and V are independent of each other and of X.

(a) Argue that knowledge of the fading factor V improves capacity by showing

$$I(X;Y|V) \geq I(X;Y).$$

(b) Incidentally, conditioning does not always increase mutual information. Give an example of $p(u,r,s)$ such that $I(U;R|S) < I(U;R)$.

4. **Erasure channel**
Let $\{\mathcal{X}, p(y|x), \mathcal{Y}\}$ be a discrete memoryless channel with capacity C. Suppose this channel is immediately cascaded with an erasure channel $\{\mathcal{Y}, p(s|y), \mathcal{S}\}$ that erases α of its symbols.



Specifically, $\mathcal{S} = \{y_1, y_2, \ldots, y_m, e\}$, and

$$\begin{aligned} \Pr\{S = y|X = x\} &= \bar{\alpha} p(y|x), \quad y \in \mathcal{Y}, \\ \Pr\{S = e|X = x\} &= \alpha. \end{aligned}$$

Determine the capacity of this channel.

5. **Random "20" questions**

Let X be uniformly distributed over $\{1, 2, \ldots, m\}$. Assume $m = 2^n$. We ask random questions: Is $X \in S_1$? Is $X \in S_2$?...until only one integer remains. All 2^m subsets S of $\{1, 2, \ldots, m\}$ are equally likely. The questions are independently and identically distributed, and subsets are drawn with replacement.

(a) Firstly, how many deterministic questions would be needed to determine X?

(b) Henceforth, we generate questions randomly in the manner described above. Without loss of generality, suppose that $X = 1$ is the random object. What is the probability that object 2 yields the same answers for k random questions as object 1?

(c) What is the expected number of objects in $\{2, 3, \ldots, m\}$ that have the same answers to k random questions as does the correct object 1?

(d) Suppose we ask $n + \sqrt{n}$ random questions. What is the expected number of wrong objects agreeing with the answers?

(e) Use Markov's inequality $\Pr\{N \geq t\} \leq \frac{EN}{t}$, to show that the probability of error (one or more wrong object remaining) goes to zero as $n \longrightarrow \infty$.