Homework Set #7

1. **Postprocessing the output.**
One is given a communication channel with transition probabilities $p(y \mid x)$ and channel capacity $C = \max_{p(x)} I(X;Y)$. A helpful statistician postprocesses the output by forming $\tilde{Y} = g(Y)$, yielding a channel $p(\tilde{y}|x)$. He claims that this will strictly improve the capacity.

   (a) Show that he is wrong.

   (b) Under what conditions does he not strictly decrease the capacity?

2. **Noisy typewriter.**
Consider a 26-key typewriter.

   (a) If pushing a key results in printing the associated letter, what is the capacity C in bits?

   (b) Now suppose that pushing a key results in printing that letter or the next (with equal probability). Thus $A \to A$ or B, and $Z \to Z$ or A. What is the capacity?

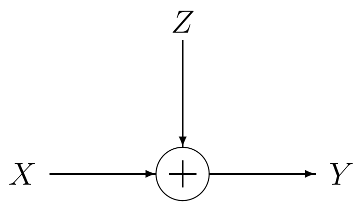
   (c) What is the highest rate code with block length one that you can find that achieves *zero* probability of error for the channel in part (b) .

3. **The Z Channel.**
The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$Q = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix} \qquad x, y \in \{0, 1\}$$

Find the capacity of the Z-channel and the maximizing input probability distribution.

4. **An additive noise channel.**
Find the channel capacity of the following discrete memoryless channel:



where $\Pr\{Z=0\} = \Pr\{Z=a\} = \frac{1}{2}$. The alphabet for x is $\mathcal{X} = \{0,1\}$. $Y = X + Z$ with real addition. Assume that Z is independent of X.

Observe that the channel capacity depends on the value of a.

5. **Channels with memory have higher capacity.**
Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$, where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0,1\}$.

Suppose that $\{Z_i\}$ has constant marginal probabilities $P\{Z_i = 1\} = p = 1 - P\{Z_i = 0\}$, but that $Z_1, Z_2, \ldots, Z_n$ are not necessarily independent. Let $C = 1 - H(p, 1-p)$. Show that $\max_{p(x_1, x_2, \ldots, x_n)} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq nC$. Comment on the implications.

6. **Channel capacity.**
Consider the channel $Y = X + Z \pmod{13}$, where

$$Z = \begin{pmatrix} 1, & 2, & 3, & 4 \\ \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4} \end{pmatrix}$$

and $X \in \{0, 1, \ldots, 12\}$.

(a) Find the capacity.

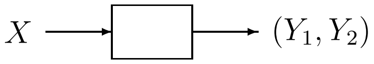
(b) What is the maximizing $p^*(x)$?

7. **Using two channels at once.**
Consider two discrete memoryless channels $(\mathcal{X}_1, p(y_1 \mid x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2 \mid x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1 \mid x_1) \times p(y_2 \mid x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, are *simultaneously* sent, resulting in y_1, y_2. Find the mutual information maximizing $p^*(x_1, x_2)$ in terms of the individual maximizing distributions $p^*(x_1)$ and $p^*(x_2)$. Find the capacity of this channel.

8. **A channel with two independent looks at Y.**
Let Y_1 and Y_2 be conditionally independent and conditionally identically distributed given X. Thus $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$.

(a) Show $I(X; Y_1, Y_2) = 2I(X; Y_1) - I(Y_1; Y_2)$.

(b) Conclude that the capacity of the channel



is less than twice the capacity of the channel

(c) How about 3 independent looks? Compare $I(X; Y_1, Y_2, Y_3)$ to $3I(X; Y_1)$.

9. **Can signal alternatives lower capacity?**
Show that adding a row to a channel transition matrix does not decrease capacity.