Solutions to Homework Set #4

1. **Horse race.**
   Suppose one is interested in maximizing the doubling rate for a horse race. Let $p_1, p_2, \ldots, p_m$ denote the win probabilities of the m horses. When do the odds $(o_1, o_2, \ldots, o_m)$ yield a higher doubling rate than the odds $(o'_1, o'_2, \ldots, o'_m)$?

   Solution: Horse race.

   Let W and W' denote the optimal doubling rates for the odds $(o_1, o_2, \ldots, o_m)$ and $(o'_1, o'_2, \ldots, o'_m)$ respectively. By Theorem 6.1.2 in the book,

$$\begin{aligned} W &= \sum p_i \log o_i - H(p), \quad \text{and} \\ W' &= \sum p_i \log o'_i - H(p) \end{aligned}$$

   where p is the probability vector $(p_1, p_2, \ldots, p_m)$. Then $W > W'$ exactly when $\sum p_i \log o_i > \sum p_i \log o'_i$; that is, when $E \log o_i > E \log o'_i$, i.e. $E \log(o_i/o'_i) > 0$.

2. **Horse race.**
   Three horses run a race. A gambler offers 3-for-1 odds on each of the horses. These are fair odds under the assumption that all horses are equally likely to win the race. The true win probabilities are known to be

$$\mathbf{p} = (p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right). \tag{1}$$

   Let $\mathbf{b} = (b_1, b_2, b_3)$, $b_i \geq 0$, $\sum b_i = 1$, be the amount invested on each of the horses. The expected log wealth is thus

$$W(\mathbf{b}) = \sum_{i=1}^{3} p_i \log 3b_i. \tag{2}$$

   (a) Maximize this over $\mathbf{b}$ to find $\mathbf{b}^*$ and W^*. Thus the wealth achieved in repeated horse races should grow to infinity like 2^{nW^*} with probability one.

(b) Show that if instead we put all of the current wealth on horse 1, the most likely winner, on each race, we will eventually go broke with probability one.

Solution: Horse race.

(a) The doubling rate

$$\begin{aligned} W(\mathbf{b}) &= \sum_i p_i \log b_i o_i \\ &= \sum_i p_i \log 3b_i \\ &= \sum p_i \log 3 + \sum p_i \log p_i - \sum p_i \log \frac{p_i}{b_i} \\ &= \log 3 - H(\mathbf{p}) - D(\mathbf{p}||\mathbf{b}) \\ &\le \log 3 - H(\mathbf{p}), \end{aligned}$$

with equality iff $\mathbf{p} = \mathbf{b}$. Hence $\mathbf{b}^* = \mathbf{p} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $W^* = \log 3 - H(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) = \frac{1}{2} \log \frac{9}{8} = 0.085$.

By the strong law of large numbers,

$$\begin{aligned} S_n &= \prod_j 3b(X_j) \\ &= 2^{n(\frac{1}{n}\sum_j \log 3b(X_j))} \\ &\to 2^{nE\log 3b(X)} \\ &= 2^{nW(\mathbf{b})} \end{aligned}$$

When $\mathbf{b} = \mathbf{b}^*$, $W(\mathbf{b}) = W^*$ and $S_n \doteq 2^{nW^*} = 2^{0.085n} = (1.06)^n$.

(b) If we put all the money on the first horse, then the probability that we do not go broke in n races is $(\frac{1}{2})^n$. Since this probability goes to zero with n, the probability of the set of outcomes where we do not ever go broke is zero, and we will go broke with probability 1.

Alternatively, if $\mathbf{b} = (1, 0, 0)$, then $W(\mathbf{b}) = -\infty$ and

$$S_n \to 2^{nW} = 0 \quad \text{w.p.1} \qquad (3)$$

by the strong law of large numbers.

3. **Horse race.**

Consider a 3-horse race with win probabilities

$$(p_1, p_2, p_3) = \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

and fair odds with respect to the (false) distribution

$$(r_1, r_2, r_3) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

Thus the odds are

$$(o_1, o_2, o_3) = (4, 4, 2) .$$

(a) What is the entropy of the race?

(b) Find the set of all bets (b_1, b_2, b_3) such that the compounded wealth in repeated plays will grow to infinity.

(c) Find the growth rate optimal b^*, and the resulting growth rate.

Solution: Horse race.

(a) The entropy of the race is given by

$$\begin{aligned} H(\mathbf{p}) &= -2(1/8)\log(1/8) - (3/4)\log(3/4) \\ &= 1.0613. \end{aligned}$$

(b) Compounded wealth will grow to infinity for the set of bets (b_1, b_2, b_3) such that $W(\mathbf{b}, \mathbf{p}) > 0$, where

$$W(\mathbf{b}, \mathbf{p}) = D(\mathbf{p}\|\mathbf{r}) - D(\mathbf{p}\|\mathbf{b}).$$

Calculating $D(\mathbf{p}\|\mathbf{r})$, this criterion becomes

$$D(\mathbf{p}\|\mathbf{b}) < 0.8137$$

We can also give an alternative expression for this criterion. First, note that we the doubling rate can be written as

$$\begin{aligned} W(\mathbf{b}, \mathbf{p}) &= \sum_{i=1}^{3} p_i \log \frac{b_i}{r_i} \\ &= \frac{3}{4}\log\frac{b_1}{r_1} + \frac{1}{8}\log\frac{b_2}{r_2} + \frac{1}{8}\log\frac{b_3}{r_3}. \end{aligned}$$

We want $W(\mathbf{b}, \mathbf{p})$ to be greater than zero,

$$0 < \frac{3}{4}\log\frac{b_1}{r_1} + \frac{1}{8}\log\frac{b_2}{r_2} + \frac{1}{8}\log\frac{b_3}{r_3}.$$

Multiplying both sides of this inequality by 8, and combining the logarithms, we have

$$0 < \log\left(\left(\frac{b_1}{r_1}\right)^6 \frac{b_2}{r_2}\frac{b_3}{r_3}\right).$$

Thus, the wealth will go to infinity for any bets (b_1, b_2, b_3) satisfying

$$b_1^6 b_2 b_3 > r_1^6 r_2 r_3 = \frac{1}{2^{15}}.$$

(c) The growth rate optimal b^* is of course given by the choice $b^* = p$, which will give us a doubling rate equal to

$$W^* = W(\mathbf{p}) = D(\mathbf{p}\|\mathbf{r}) = 0.8137.$$

4. **Entropy of a fair horse race.**
Let $X \sim p(x)$, $x = 1, 2, \ldots, m$, denote the winner of a horse race. Suppose the odds $o(x)$ are fair with respect to $p(x)$, i.e., $o(x) = \frac{1}{p(x)}$. Let $b(x)$ be the amount bet on horse x, $b(x) \geq 0$, $\sum_1^m b(x) = 1$. Then the resulting wealth factor is $S(x) = b(x)o(x)$, with probability $p(x)$.

(a) Find the expected wealth $ES(X)$.

(b) Find W^*, the optimal growth rate of wealth.

(c) Suppose

$$Y = \begin{cases} 1, & X = 1 \text{ or } 2 \\ 0, & \text{otherwise} \end{cases}$$

If this side information is available before the bet, how much does it increase the growth rate W^*?

(d) Find $I(X;Y)$.

Entropy of a fair horse race.

(a) The expected wealth $ES(X)$ is

$$\begin{aligned} ES(X) &= \sum_{x=1}^{m} S(x)p(x) && (4)\\ &= \sum_{x=1}^{m} b(x)o(x)p(x) && (5)\\ &= \sum_{x=1}^{m} b(x), \quad (\text{since } o(x) = 1/p(x)) && (6)\\ &= 1. && (7) \end{aligned}$$

(b) The optimal growth rate of wealth, W^*, is achieved when $b(x) = p(x)$ for all x, in which case,

$$\begin{aligned} W^* &= E(\log S(X)) && (8)\\ &= \sum_{x=1}^{m} p(x)\log(b(x)o(x)) && (9)\\ &= \sum_{x=1}^{m} p(x)\log(p(x)/p(x)) && (10)\\ &= \sum_{x=1}^{m} p(x)\log(1) && (11)\\ &= 0, && (12)\end{aligned}$$

so we maintain our current wealth.

(c) The increase in our growth rate due to the side information is given by $I(X;Y)$. Let $q = \Pr(Y = 1) = p(1) + p(2)$.

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) && (13)\\ &= H(Y) \quad \text{(since } Y \text{ is a deterministic function of } X\text{)} && (14)\\ &= H(q). && (15)\end{aligned}$$

(d) Already computed above.

5. **Negative horse race.**
Consider a horse race with m horses with win probabilities $p_1, p_2, \ldots, p_m$. Here the gambler hopes a given horse will lose. He places bets $(b_1, b_2, \ldots, b_m)$, $\sum_{i=1}^{m} b_i = 1$, on the horses, loses his bet b_i if horse i wins, and retains the rest of his bets. (No odds.) Thus $S = \sum_{j \neq i} b_j$, with probability p_i, and one wishes to maximize $\sum p_i \ln(1 - b_i)$ subject to the constraint $\sum b_i = 1$.

(a) Find the growth rate optimal investment strategy b^*. Do *not* constrain the bets to be positive, but do constrain the bets to sum to 1. (This effectively allows short selling and margin.)

(b) What is the optimal growth rate?

Solution: Negative horse race.
To find the constrained maximum we want to maximize the following Lagrangian over $\mathbf{b}$.

$$L(\lambda, \mathbf{b}) = \sum p_i \ln(1 - b_i) + \lambda \sum b_i$$

Taking the partial derivative with respect to b_i and setting it equal to zero gives:

$$\frac{\partial L(\lambda, \mathbf{b})}{\partial b_i} = \frac{-p_1}{1-b_i} + \lambda = 0$$

Solving for b_i gives $b_i = 1 - \frac{p_i}{\lambda}$. Further, solving the constraint

$$1 = \sum b_i = \sum 1 - \frac{p_i}{\lambda} = m - \frac{1}{\lambda}$$

gives $\lambda = \frac{1}{m-1}$. Thus

$$b_i^* = 1 - (m-1)p_i,$$

and

$$W^* = \sum p_i \ln((m-1)p_i) = \ln(m-1) - H(\mathbf{p}).$$

6. **The St. Petersburg paradox.**
Many years ago in ancient Leningrad the following gambling proposition caused great consternation. For an entry fee of c units, a gambler receives a payoff of 2^k units with probability $2^{-k}, k = 1, 2, \ldots$.

(a) Show that the expected payoff for this game is infinite. For this reason, it was argued that $c = \infty$ was a "fair" price to pay to play this game. Most people find this answer absurd.

(b) Suppose that the gambler can buy a share of the game. For example, if he invests $c/2$ units in the game, he receives $1/2$ a share and a return $X/2$, where $\Pr(X = 2^k) = 2^{-k}, k = 1, 2, \ldots$. Suppose $X_1, X_2, \ldots$ are i.i.d. according to this distribution and the gambler reinvests all his wealth each time. Thus his wealth S_n at time n is given by

$$S_n = \prod_{i=1}^{n} \left(\frac{X_i}{c} \right). \qquad (16)$$

Show that this limit is ∞ or 0, with probability one, accordingly as $c < c^*$ or $c > c^*$. Identify the "fair" entry fee c^*.

More realistically, the gambler should be allowed to keep a proportion $\overline{b} = 1 - b$ of his money in his pocket and invest the rest in the St. Petersburg game. His wealth at time n is then

$$S_n = \prod_{i=1}^{n} \left(\overline{b} + \frac{bX_i}{c} \right). \qquad (17)$$

Let

$$W(b, c) = \sum_{k=1}^{\infty} 2^{-k} \log \left(1 - b + \frac{b2^k}{c} \right). \qquad (18)$$

We have

$$S_n \doteq 2^{nW(b,c)} \tag{19}$$

Let

$$W^*(c) = \max_{0 \le b \le 1} W(b,c) \tag{20}$$

be the optimal growth rate and let b^* be the corresponding investment in St. Petersburg.

(c) For what value of the entry fee c does the optimizing value b^* drop below 1?

Note that since $W^*(c) > 0$, for all c , we can conclude that any entry fee c is fair.

Solution: The St. Petersburg paradox.

(a) The expected return,

$$EX = \sum_{k=1}^{\infty} p(X = 2^k) 2^k = \sum_{k=1}^{\infty} 2^{-k} 2^k = \sum_{k=1}^{\infty} 1 = \infty. \tag{21}$$

Thus the expected return on the game is infinite.

(b) By the strong law of large numbers, we see that

$$\frac{1}{n} \log S_n = \frac{1}{n} \sum_{i=1}^{n} \log X_i - \log c \to E \log X - \log c, \text{ w.p.1} \tag{22}$$

and therefore S_n goes to infinity or 0 according to whether $E \log X$ is greater or less than $\log c$. Therefore

$$\log c^* = E \log X = \sum_{k=1}^{\infty} k 2^{-k} = 2. \tag{23}$$

Therefore a fair entry fee is 4 units if the gambler is forced to invest all his money.

(c) If the gambler is not required to invest all his money, then the growth rate is

$$W(b, c) = \sum_{k=1}^{\infty} 2^{-k} \log \left(1 - b + \frac{b 2^k}{c} \right). \tag{24}$$

For $b = 0$, $W = 0$, and for $b = 1$, $W = E \log X - \log c = 2 - \log c$. Differentiating to find the optimum value of b, we obtain

$$\frac{\partial W(b, c)}{\partial b} = \sum_{k=1}^{\infty} 2^{-k} \frac{1}{\left(1 - b + \frac{b 2^k}{c} \right)} \left(-1 + \frac{2^k}{c} \right) \tag{25}$$

Unfortunately, there is no explicit solution for the b that maximizes W for a given value of c, and we have to solve this numerically on the computer.

We have illustrated the results with four plots. The first (Figure 2) shows $W(b, c)$ as a function of b and c. The second (Figure 3) shows b^* as a function of c. The third (Figure 4) and the fourth (Figure 5) show W^* and 2^{W^*} as a function of c.

From Figure 3, it is clear that b^* is less than 1 for $c > 3$. We can also see this analytically by calculating the slope $\frac{\partial W(b,c)}{\partial b}$ at $b = 1$. First observe that $W(b, c)$

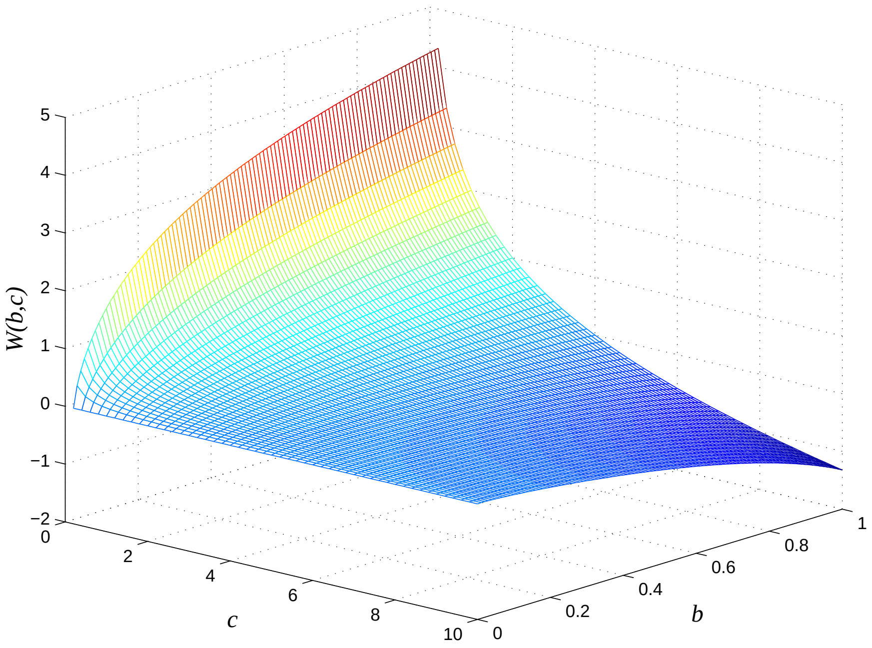

Figure 1: $W(b, c)$ as a function of b and c.

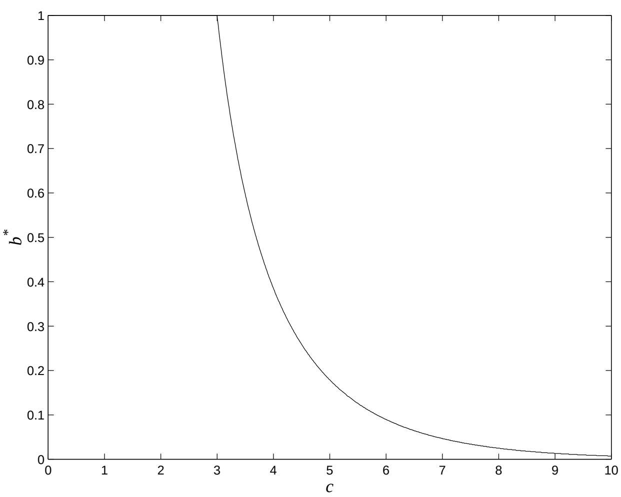

Figure 2: b^* as a function of c.

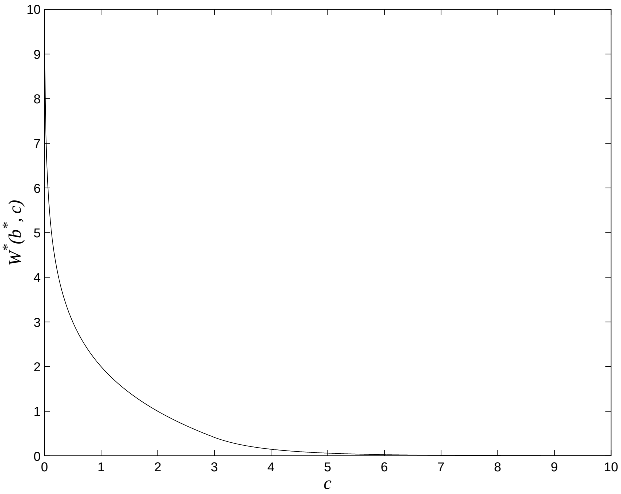

Figure 3: $W^*(b^*, c)$ as a function of c.

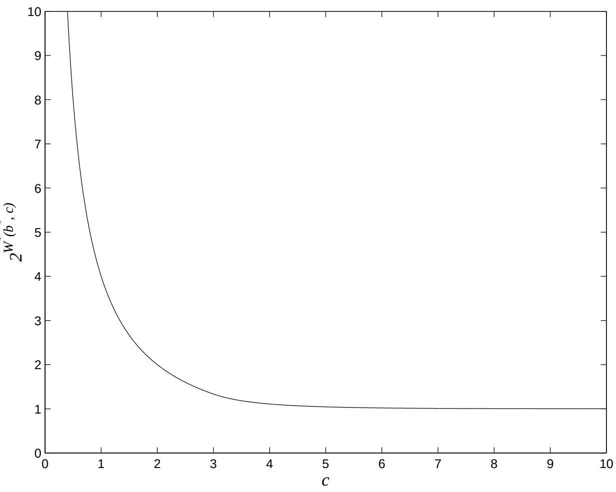

Figure 4: $2^{W^*(b^*,c)}$ as a function of c.

is a strictly concave function of b given c, since

$$\begin{aligned}
\frac{\partial^2 W(b,c)}{\partial b^2} &= \frac{\partial}{\partial b}\frac{\partial W(b,c)}{\partial b} \\
&= \frac{\partial}{\partial b}\left[\sum_{k=1}^{\infty} 2^{-k} \frac{1}{\left(1-b+\frac{b2^k}{c}\right)}\left(-1+\frac{2^k}{c}\right)\right] \\
&= \sum_{k=1}^{\infty} 2^{-k} \frac{-1}{\left(1-b+\frac{b2^k}{c}\right)^2}(-1+\frac{2^k}{c})^2 \\
&< 0
\end{aligned}$$

Now we can evaluate $\frac{\partial W(b,c)}{\partial b}$ at $b=1$ using Equation (25):

$$\begin{aligned}
\frac{\partial W(b=1,c)}{\partial b} &= \sum_k \frac{2^{-k}}{\frac{2^k}{c}}\left(\frac{2^k}{d}-1\right) \\
&= \sum_{k=1}^{\infty} 2^{-k} - \sum_{k=1}^{\infty} c2^{-2k} \\
&= 1-\frac{c}{3},
\end{aligned}$$

which is positive for $c < 3$. Thus for $c < 3$, the optimal value of b lies on the boundary of the region of b's, and for $c > 3$, the optimal value of b lies in the interior.

Finally note that $W^*(c) > 0$ for all c since $\frac{\partial W(b=0,c)}{\partial b} = \sum_k -2^{-k} + \frac{1}{c} = \infty$, $W = 0$ when $b = 0$, and W is concave.

7. **Gambling.**

Suppose a gamble X is available that will turn each unit of investment into X units. Let

$$X = \begin{cases} 3, & \text{with probability } 1/2 \\ 0, & \text{with probability } 1/2. \end{cases}$$

Thus, with full reinvestment each time, the wealth S_n at the end of n i.i.d. investments will be

$$S_n = \prod_{i=1}^{n} X_i.$$

(a) Find EX and ES_n.

(b) Show that $S_n \to 0$ in probability; that is, show, for $\epsilon > 0$,

$$\Pr\{S_n \geq \epsilon\} \to 0,$$

as $n \to \infty$.

(c) The gambler decides not to bet the whole amount each time. He invests a proportion b of his current wealth in X and keeps the rest in cash, thus resulting in a wealth factor of $(1-b+bX)$, and a wealth S_n at time n of

$$S_n = \prod_{i=1}^{n} (1-b+bX_i).$$

Find the allocation b^* that maximizes the growth rate of wealth.

(d) Find

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n$$

for this choice of b.

Solution: Gambling.

(a)

$$\mathbf{E}X = 3 \times \frac{1}{2} + 0 \times \frac{1}{2} = \frac{3}{2}$$

$$\mathbf{E}S_n = \mathbf{E}\prod_{i=1}^{n} X_i = \prod_{i=1}^{n} \mathbf{E}X_i = \left(\frac{3}{2}\right)^n$$

(b) Observe that S_n is greater than zero if and only if $X_i = 3$ for all $i \leq n$. So

$$\begin{aligned}\Pr\{S_n \geq \epsilon\} &\leq \Pr\{S_n > 0\} \\ &= \left(\frac{1}{2}\right)^n \to 0\end{aligned}$$

(c) The growth rate of wealth is maximized by maximizing the expected log.

$$\begin{aligned}\mathbf{E}\ln(1-b+bX) &= \frac{1}{2}\ln(1-b+3b) + \frac{1}{2}\ln(1-b) \\ &= \frac{1}{2}\ln(1+2b) + \frac{1}{2}\ln(1-b)\end{aligned}$$

Taking the derivative with respect to b gives:

$$\frac{d(\frac{1}{2}\ln(1+2b) + \frac{1}{2}\ln(1-b))}{db} = \frac{2}{2(1+2b)} + \frac{-1}{2(1-b)}$$

Setting the derivative equal to zero and solving for b gives:

$$b^* = \frac{1}{4}.$$

(d)

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \ln S_n &= \mathbf{E} \ln((1-b) + bX) \\
&= \frac{1}{2} \ln(1 + 2(1/4)) + \frac{1}{2} \ln(1 - (1/4)) \\
&= \frac{1}{2} \ln(9/8) \\
&= 0.0589
\end{aligned}$$

Thus $S_n \doteq e^{0.0589n}$.