

### Homework Set #4

1. **Horse race.**

Suppose one is interested in maximizing the doubling rate for a horse race. Let  $p_1, p_2, \dots, p_m$  denote the win probabilities of the  $m$  horses. When do the odds  $(o_1, o_2, \dots, o_m)$  yield a higher doubling rate than the odds  $(o'_1, o'_2, \dots, o'_m)$ ?

2. **Horse race.**

Three horses run a race. A gambler offers 3-for-1 odds on each of the horses. These are fair odds under the assumption that all horses are equally likely to win the race. The true win probabilities are known to be

$$\mathbf{p} = (p_1, p_2, p_3) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right). \quad (1)$$

Let  $\mathbf{b} = (b_1, b_2, b_3)$ ,  $b_i \geq 0$ ,  $\sum b_i = 1$ , be the amount invested on each of the horses. The expected log wealth is thus

$$W(\mathbf{b}) = \sum_{i=1}^3 p_i \log 3b_i. \quad (2)$$

- (a) Maximize this over  $\mathbf{b}$  to find  $\mathbf{b}^*$  and  $W^*$ . Thus the wealth achieved in repeated horse races should grow to infinity like  $2^{nW^*}$  with probability one.
- (b) Show that if instead we put all of the current wealth on horse 1, the most likely winner, on each race, we will eventually go broke with probability one.

3. **Horse race.**

Consider a 3-horse race with win probabilities

$$(p_1, p_2, p_3) = \left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$$

and fair odds with respect to the (false) distribution

$$(r_1, r_2, r_3) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$$

Thus the odds are

$$(o_1, o_2, o_3) = (4, 4, 2).$$

- (a) What is the entropy of the race?
- (b) Find the set of all bets  $(b_1, b_2, b_3)$  such that the compounded wealth in repeated plays will grow to infinity.
- (c) Find the growth rate optimal  $b^*$ , and the resulting growth rate  $W^*$ .

#### 4. Entropy of a fair horse race.

Let  $X \sim p(x)$ ,  $x = 1, 2, \dots, m$ , denote the winner of a horse race. Suppose the odds  $o(x)$  are fair with respect to  $p(x)$ , i.e.,  $o(x) = \frac{1}{p(x)}$ . Let  $b(x)$  be the amount bet on horse  $x$ ,  $b(x) \geq 0$ ,  $\sum_1^m b(x) = 1$ . Then the resulting wealth factor is  $S(x) = b(x)o(x)$ , with probability  $p(x)$ .

- (a) Find the expected wealth  $ES(X)$ .
- (b) Find  $W^*$ , the optimal growth rate of wealth.
- (c) Suppose

$$Y = \begin{cases} 1, & X = 1 \text{ or } 2 \\ 0, & \text{otherwise} \end{cases}$$

If this side information is available before the bet, how much does it increase the growth rate  $W^*$ ?

- (d) Find  $I(X; Y)$ .

#### 5. Negative horse race.

Consider a horse race with  $m$  horses with win probabilities  $p_1, p_2, \dots, p_m$ . Here the gambler hopes a given horse will lose. He places bets  $(b_1, b_2, \dots, b_m)$ ,  $\sum_{i=1}^m b_i = 1$ , on the horses, loses his bet  $b_i$  if horse  $i$  wins, and retains the rest of his bets. (No odds.) Thus  $S = \sum_{j \neq i} b_j$ , with probability  $p_i$ , and one wishes to maximize  $\sum p_i \ln(1 - b_i)$  subject to the constraint  $\sum b_i = 1$ .

- (a) Find the growth rate optimal investment strategy  $b^*$ . Do *not* constrain the bets to be positive, but do constrain the bets to sum to 1. (This effectively allows short selling and margin.)
- (b) What is the optimal growth rate?

## 6. The St. Petersburg paradox.

Many years ago in ancient St. Petersburg the following gambling proposition caused great consternation. For an entry fee of  $c$  units, a gambler receives a payoff of  $2^k$  units with probability  $2^{-k}$ ,  $k = 1, 2, \dots$ .

- (a) Show that the expected payoff for this game is infinite. For this reason, it was argued that  $c = \infty$  was a “fair” price to pay to play this game. Most people find this answer absurd.
- (b) Suppose that the gambler can buy a share of the game. For example, if he invests  $c/2$  units in the game, he receives  $1/2$  a share and a return  $X/2$ , where  $\Pr(X = 2^k) = 2^{-k}$ ,  $k = 1, 2, \dots$ . Suppose  $X_1, X_2, \dots$  are i.i.d. according to this distribution and the gambler reinvests all his wealth each time. Thus his wealth  $S_n$  at time  $n$  is given by

$$S_n = \prod_{i=1}^n \left( \frac{X_i}{c} \right). \quad (3)$$

Show that this limit is  $\infty$  or  $0$ , with probability one, accordingly as  $c < c^*$  or  $c > c^*$ . Identify the “fair” entry fee  $c^*$ .

More realistically, the gambler should be allowed to keep a proportion  $\bar{b} = 1 - b$  of his money in his pocket and invest the rest in the St. Petersburg game. His wealth at time  $n$  is then

$$S_n = \prod_{i=1}^n \left( \bar{b} + \frac{bX_i}{c} \right). \quad (4)$$

Let

$$W(b, c) = \sum_{k=1}^{\infty} 2^{-k} \log \left( 1 - b + \frac{b2^k}{c} \right). \quad (5)$$

We have

$$S_n \doteq 2^{nW(b,c)} \quad (6)$$

Let

$$W^*(c) = \max_{0 \leq b < 1} W(b, c) \quad (7)$$

be the optimal growth rate and let  $b^*$  be the corresponding investment in St. Petersburg.

- (c) For what value of the entry fee  $c$  does the optimizing value  $b^*$  drop below 1?

Note that since  $W^*(c) > 0$ , for all  $c$ , we can conclude that any entry fee  $c$  is fair. However, for even moderate values of  $c$ ,  $c = 6$  say, we will choose to invest very little in St. Petersburg, and the result won't change our net worth by much. This is a possible resolution of the paradox.

## 7. Gambling.

Suppose a gamble  $X$  is available that will turn each unit of investment into  $X$  units. Let

$$X = \begin{cases} 3, & \text{with probability } 1/2 \\ 0, & \text{with probability } 1/2. \end{cases}$$

Thus, with full reinvestment each time, the wealth  $S_n$  at the end of  $n$  i.i.d. investments will be

$$S_n = \prod_{i=1}^n X_i.$$

- (a) Find  $EX$  and  $ES_n$ .
- (b) Show that  $S_n \rightarrow 0$  in probability; that is, show, for  $\epsilon > 0$ ,

$$\Pr\{S_n \geq \epsilon\} \rightarrow 0,$$

as  $n \rightarrow \infty$ .

- (c) The gambler decides not to bet the whole amount each time. He invests a proportion  $b$  of his current wealth in  $X$  and keeps the rest in cash, thus resulting in a wealth factor of  $(1 - b + bX)$ , and a wealth  $S_n$  at time  $n$  of

$$S_n = \prod_{i=1}^n (1 - b + bX_i).$$

Find the allocation  $b^*$  that maximizes the growth rate of wealth.

- (d) Find

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln S_n$$

for this choice of  $b$ .