EE 376A/Stat 376A
Information Theory
Prof. T. Cover

Handout #9
Thursday, January 22, 2009
Due Thursday, January 29, 2009

Homework Set #3

1. **Random walk in a cube.**
   A bird flies from room to room in a $3 \times 3 \times 3$ cube (equally likely through each interior wall). What is the entropy rate?

2. **Entropy of graphs.**
   Consider a random walk on a (connected) graph with 3 edges.
   (a) Which graph has the lowest entropy rate? What is the rate?
   (b) Which has the highest entropy rate?

3. **Stationary processes.**
   Let $\ldots, X_{-1}, X_0, X_1, \ldots$ be a stationary (not necessarily Markov) stochastic process. Which of the following statements are true? Prove or provide a counterexample.
   (a) $H(X_n|X_0) = H(X_{-n}|X_0)$.
   (b) $H(X_n|X_0) \geq H(X_{n-1}|X_0)$.
   (c) $H(X_n|X_1^{n-1}, X_{n+1})$ is nonincreasing in n.
   (d) $H(X_{n+1}|X_1^n, X_{n+2}^{2n+1})$ is nonincreasing in n.

4. **Entropy rate.**
   Let $\{X_i\}$ be a stationary $\{0,1\}$-valued stochastic process obeying
   $$X_{k+1} = X_k \oplus X_{k-1} \oplus Z_{k+1}$$
   where $\{Z_i\}$ is Bernoulli(p) and $\oplus$ denotes mod 2 addition. What is the entropy rate $H(\mathcal{X})$?

5. **The past has little to say about the future.**
   For a stationary stochastic process $X_1, X_2, \ldots,$ show that
   $$\lim_{n\to\infty} \frac{1}{2n} I(X_1, X_2, \ldots, X_n; X_{n+1}, X_{n+2}, \ldots, X_{2n}) = 0.$$

Thus the dependence between adjacent n-blocks of a stationary process does not grow linearly with n.

6. **Markov chain.**

$$P = [P_{ij}] = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Let X_1 be uniformly distributed over the states $\{0, 1, 2\}$. Let $\{X_i\}_1^\infty$ be a Markov chain with transition matrix P, thus $P(X_{n+1} = j | X_n = i) = P_{ij}, i, j \in \{0, 1, 2\}$.

(a) Is $\{X_n\}$ stationary?

(b) Find $\lim_{n\to\infty} \frac{1}{n} H(X_1, \ldots, X_n)$.

Now consider the derived process $Z_1, Z_2, \ldots, Z_n$, where

$$\begin{aligned} Z_1 &= X_1 \\ Z_i &= X_i - X_{i-1} \pmod 3, \qquad i = 2, \ldots, n. \end{aligned}$$

Thus Z^n encodes the transitions, not the states.

(c) Find $H(Z_1, Z_2, ..., Z_n)$.

(d) Find $H(Z_n)$ and $H(X_n)$, for $n \geq 2$.

(e) Find $H(Z_n | Z_{n-1})$ for $n \geq 2$.

(f) Are Z_{n-1} and Z_n independent for $n \geq 2$?

7. **Waiting times.**
Let X be the waiting time for the first heads to appear in successive flips of a fair coin. For example, $\Pr\{X = 3\} = (\frac{1}{2})^3$. Let S_n be the waiting time for the nth head to appear. Thus,

$$\begin{aligned} S_0 &= 0 \\ S_{n+1} &= S_n + X_{n+1} \end{aligned}$$

where $X_1, X_2, X_3, \ldots$ are i.i.d according to the distribution above.

(a) Is the process $\{S_n\}$ stationary?

(b) Calculate $H(S_1, S_2, \ldots, S_n)$.

(c) Does the process $\{S_n\}$ have an entropy rate? If so, what is it? If not, why not?

(d) What is the expected number of fair coin flips required to generate a random variable having the same distribution as S_n?

8. **Entropy rate.**
Let $\{X_i\}$ be a stationary stochastic process with entropy rate $H(\mathcal{X})$.

(a) Argue that $H(\mathcal{X}) \leq H(X_1)$.

(b) What are the conditions for equality?