EE 372
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Quantization, Compression, and Classification

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Partition subcodes and supercodes

A quantizer q determined by $\mathcal{S}$ is a *partition subcode* of the quantizer q' determined by $\mathcal{S}'$ if $\mathcal{S} \subset \mathcal{S}'$. (The Lloyd conditions then imply the corresponding decoder and weighting)

Similarly, q' is a *partition supercode* of q.

Lemma 4. *If $\mathcal{S} \subset \mathcal{S}'$, then*

$$\begin{aligned} D(\mathcal{S}) &\geq D(\mathcal{S}') \\ H(\mathcal{S}) &\leq H(\mathcal{S}') \\ N(\mathcal{S}) &\leq N(\mathcal{S}') \\ R_\eta(\mathcal{S}) &\leq R_\eta(\mathcal{S}') \end{aligned}$$

where they appropriate optimal weighted codebook for each partition is assumed.

A partition subcode (supercode) will have larger (smaller) average distortion and larger (smaller) average rate, but the Lagrangian distortion might increase or decrease depending on the ratio of the change and λ.

Weighted codebook subcodes and supercodes

weighted codebooks $(\mathcal{D}, w)$ $(\mathcal{D}', w')$ with index sets $\mathcal{I} \subset \mathcal{I}'$

$(\mathcal{D}, w)$ is a *codebook subcode* of $(\mathcal{D}', w')$ ($(\mathcal{D}', w')$ is a *codebook supercode* of $(\mathcal{D}, w)$) and write $(\mathcal{D}, w) \subset (\mathcal{D}', w')$ if

- the reproduction codebook of the subcode is a subset of the reproduction codebook of the larger code: $\{\mathcal{D}(i), i : w(i) > 0\} \subset \{\mathcal{D}'(i), i : w'(i) > 0\}$ and the codewords are ordered so that that $\mathcal{D}(i) = \mathcal{D}'(i)$ for all $i : w(i) > 0$.
- Define $J = \{i : w'(i) > 0, w(i) = 0\}$, the set of all indices corresponding to reproduction codewords removed from the larger code. Then $w'(i) = \alpha w(i)$ for all $i \notin J$ for some $\alpha \in (0, 1]$.

Note: Relative weights for common codewords unchanged.

Can use $\alpha = 1$ for pruning existing codebook, but could violate subpmf condition if want to grow codebook with w already a pmf.

Can find subcodebook by using list encoding.

Large literature exists for growing and pruning codes based on partitions – tree-structured vector quantization.

Relatively little done for growing and pruning codes based on weighted codebooks. Similar ideas arise in agglomorative and conglomorative clustering algorithms.

Weighted codebook sub and supercodes have similar behavior to partition sub and supercodes in terms of distortion and rate, but not quite the same:

$(\mathcal{D}, w), (\mathcal{D}', w')$, $\mathcal{S} = \{S_i\}$ and $\mathcal{S}' = \{S'_i\}$ corresponding Lloyd optimal encoder partitions.

$\mathcal{I}' = \{i : w'(i) > 0\}$, $J = \{i : i \in \mathcal{I}', w(i) = 0\}$

If $x \in S'_i$ then

$$d(x, \mathcal{D}'(i)) - \lambda(1-\eta)\ln w'(i) \leq d(x, \mathcal{D}'(j)) - \lambda(1-\eta)\ln w'(j), \text{ all } j \neq i$$

If $i, j \notin J$, then also

$$d(x, \mathcal{D}(i)) - \lambda(1-\eta)\ln w(i) \leq d(x, \mathcal{D}(j)) - \lambda(1-\eta)\ln w(j) \Rightarrow x \in S_i \Rightarrow S'_i \subset S_i$$

For each $j \in J$, define $S_{i,j} = S'_j \cap S_i$ (part of removed atom S'_j put into S_i)

$$\begin{aligned} S_i &= S'_i \cup \bigcup_{j \in J} S_{i,j}; i \notin J \\ S'_j &= \bigcup_{i \notin J} S_{i,j}; j \in J \\ S'_i &= S_i - \bigcup_{j \in J} S_{i,j}; i \notin J \end{aligned}$$

Let $p'(i) = P(S'_i)$, $p(i) = P(S_i)$ and observe that for $i \notin J$, $p(i) \geq p'(i)$.

Rate Obviously $N(w) \leq N(w')$. It also follows easily that

$$\begin{aligned} -\sum_i p'(i)\ln w'(i) &= -\sum_{i \notin J} p'(i)\ln w'(i) - \sum_{i \in J} p'(i)\ln w'(i) \\ &\geq -\sum_{i \notin J} p'(i)\ln w'(i) \\ &\geq -\sum_i p(i)\ln w(i). \end{aligned} \tag{28}$$

Thus,

$$R_\eta(\mathcal{E}', w') \geq R_\eta(\mathcal{E}, w), \tag{29}$$

as was the case with partition sub and supercodes.

Average distortion

$$\begin{aligned} D(\mathcal{E},\mathcal{D},w) &= \sum_{i\in\mathcal{I}'-J}\int_{S_i} d(x,\mathrm{cent}(S_i))\,dP(x) \\ &= \sum_{i\in\mathcal{I}'-J}\int_{S_i'\cup\bigcup_{j\in J}S_{i,j}} d(x,\mathrm{cent}(S_i))\,dP(x) \\ &= \sum_{i\in\mathcal{I}'-J}\int_{S_i'} d(x,\mathrm{cent}(S_i))\,dP(x)+ \\ &\quad \sum_{i\in\mathcal{I}'-J}\int_{\bigcup_{j\in J}S_{i,j}} d(x,\mathrm{cent}(S_i))\,dP(x). \end{aligned}$$

Unfortunately this does not imply that $D(q)\geq D(q')$ in general. If, however $\alpha=1$, then the implication does follow:

$$\min_i d(x,\mathcal{D}(i))-\lambda(1-\eta)\ln w(i)\geq \min_i d(x,\mathcal{D}'(i))-\lambda(1-\eta)\ln w'(i)$$

and hence

$$D(q)-\lambda(1-\eta)\sum_i p(i)\ln w(i)\geq D(q')-\lambda(1-\eta)\sum_i p'(i)\ln w'(i)$$

which with (28) implies that $D(q)\geq D(q')$. Thus

Lemma 5. *If q is a subcode of q' with $\alpha=1$, then*

$$\begin{aligned} D(q) &\geq D(q') \\ R_\eta(q) &\leq R_\eta(q') \end{aligned}$$

that is, subcodes have smaller distortion and larger rate.

Codebook size If a quantizer q is optimal there can be no subcode or supercode q' for which $D(q')+\lambda R(q')<D(q)+\lambda R(q)$. pruning/growing

Growing and pruning can be used either for improving a code at a fixed λ, or finding codes for smaller or larger λ that optimize the distortion/rate tradeoff.

Step 0: Initialization Given initial $(\mathcal{D}_0,w_0)$.
Compute $\rho_0=\rho(\mathcal{S}(\mathcal{D}_0,w_0),\mathcal{D}_0,w_0)$. Set $m=1$

Step 1: Partition improvement Given$(\mathcal{D}_{m-1},w_{m-1})$, form an optimum partition $\mathcal{S}_m=\mathcal{S}(\mathcal{D}_{m-1},w_{m-1})$.

Step 2: Weighted codebook improvement Given the partition $\mathcal{S}_m$, form an optimum weighted codebook $(\mathcal{D}_m,w_m)=(\mathcal{D}(\mathcal{S}_m),w(\mathcal{S}_m))$. Compute $\rho_m=\rho(\mathcal{S}_m,\mathcal{D}_m,w_m)$.

Step 3: Test Test $\rho_{m-1}-\rho_m$. If small enough, go to Step 4. Else set $m=m+1$, go to Step 1.

Step 4: Grow/Prune Test sub/super codes for possible improvement for fixed λ or for changed λ. Quit or go to step 1.

Shannon rate-distortion theory

Shannon (1949, 1959)

Branch of information theory: source coding subject to a fidelity criterion

Information measures

So far:

X is a random vector with distribution P

α is a quantizer or a quantizer encoder

Then

$$H(\alpha(X)) \triangleq \sum_{i \in \mathcal{I}} P(\alpha(X) = i) \ln \frac{1}{P(\alpha(X) = i)}.$$

General definition of entropy for a random vector (Komogorov-Sinai):

$$H(X) \triangleq \sup_{\alpha} H(\alpha(X))$$

supremum over all quantizers

If X continuous, $H(X) = \infty$ (except for trivial cases)

Given two distributions P and P' describing a random variable X and a quantizer α, the relative-entropy or Kullback Leibler divergence is

$$H_{P||P'}(\alpha(X)) \triangleq \sum_{i \in \mathcal{I}} P(\alpha(X) = i) \ln \frac{P(\alpha(X) = i)}{P'(\alpha(X) = i)}$$

Relative entropy of the random vector X

$$H(P||P') \triangleq \sup_{\alpha} H_{P||P'}(\alpha(X)).$$

If P and P' are discrete with pmf's f and g, then

$$H(P||P') = H(f||g) = \sum_i f_i \ln \frac{f_i}{g_i}.$$

If P and P' are determined by densities f and g, then

$$H(P||P') = H(f||g) = \int dx\, f(x) \ln \frac{f(x)}{g(x)}$$

Average mutual information between quantized random vectors X and Y:

$$I(\alpha(X), \beta(Y)) \triangleq H(\alpha(X)) + H(\beta(Y)) - H(\alpha(X), \beta(Y))$$

mutual information between the random vectors by

$$I(X, Y) \triangleq \sup_{\alpha, \beta} I(\alpha(X), \beta(Y)).$$

Alternatively,

$$I(\alpha(X), \beta(Y)) = H_{P_{\alpha(X),\beta(Y)}||P_{\alpha(X)} \times P_{\beta(Y)}}(\alpha(X), \beta(Y)),$$

and

$$I(X, Y) = H_{P_{X,Y}||P_X \times P_Y}.$$

It is straightforward to prove the following useful inequalities:

$$\begin{aligned} I(\alpha(X),\beta(Y)) &\geq 0 \\ H(\alpha(X)) &\leq H(\alpha(X),\beta(X)) \\ I(\alpha(X),\beta(Y)) &\leq H(\beta(Y)) \\ I(X,Y) &\leq H(Y) \end{aligned}$$

For example, the first two follow from the divergence inequality and the third follows from the second and the definition.

A lower bound to average distortion

X k-dimensional random vector of samples from a stationary source.

Quantize by a quantizer $q = (\mathcal{E}, \mathcal{D}, \ell)$.

Focus on distortion-rate formulation variation on Shannon's rate-distortion formulation

Shannon considered fixed-rate codes of vectors, or *block codes*

But lower bound works for all cases (all $\eta \in [0,1]$)

Fix $\eta \in [0,1]$, $R \geq 0$ and assume $R_\eta(\mathcal{E}, w) \leq R$

From the basic information measure inequalities and Corollary 1,

$$\begin{aligned} R_\eta(\mathcal{E}, w) &= (1-\eta)\sum_i p(i)\ln\frac{1}{w_i} + \eta \ln N(w) \\ &\geq (1-\eta)H(\mathcal{E}(X)) + \eta \ln N(w) \\ &\geq H(\mathcal{E}(X)) \geq H(\mathcal{D}(\mathcal{E}(X))) \\ &\geq I(X;\mathcal{D}(\mathcal{E}(X))). \end{aligned}$$

Thus for any code $q = (\mathcal{E}, \mathcal{D}, p)$ with $R_\eta(\mathcal{E}, w) \leq R$,

$$\begin{aligned} D(q) &= E[d(X,\hat{X})] \\ &\geq \inf_{f_{Y|X}: I(X;Y)\leq R} E[d(X,Y)] \triangleq D_k(R), \end{aligned}$$

Shannon's *distortion-rate function (DRF)* for $X = X^k$.

Since true for *all* quantizers, $\delta_\eta(R) \geq D_k(R)$ all $\eta \in [0,1]$

Shannon DRF's are not in general easy to compute, but

- there is a further lower bound, the *Shannon lower bound*, which is easy to compute and provides a general and useful, if conservative, lower bound to average distortion.
- In some cases, such as Gaussian processes, the Shannon DRF can be explicitly evaluated
- Arimoto-Blahut algorithm for numerical evaluation

Lower bound is a "negative" or "converse" result – can do no better, there is also a positive theorem, but requires limits of large dimension k

Fix R and normalize rate and distortion by dimension. Define

$$\delta_\eta^{(k)}(R) = \frac{\delta_\eta(f_{X^k}, kR)}{k}$$

$$\overline{\delta}_\eta(R) = \inf_k \delta_\eta^{(k)}(R),$$

Then Shannon distortion rate function for the process $\{X_n\}$ is

$$\overline{\delta}_\eta(R) \geq \inf_k \frac{1}{k} D_k(kR) \triangleq \overline{D}(R).$$

For stationary processes, infima are limits as $k \to \infty$

Positive coding theorem

$$\overline{\delta}_\eta(R) = \overline{D}(R).$$

Much harder to prove. Existence proof, need asymptotically large dimension.

Result holds regardless of η and Shannon DRF has no η in its formulation. For large dimension, optimizing for fixed or variable rate codes makes no difference — both have the same limiting performance!

The Shannon lower bound

Shannon (1959)

For any conditional pdf $f_{Y|X}$ (“test channel”) define the Lagrangian average distortion

$$\begin{aligned} \rho(\lambda, f_{Y|X}) &= E[d(X,Y)] + \lambda I(X;Y) \\ &= \int dx \int dy f_{Y|X}(y|x) f_X(x) \times \\ &\quad \left(||x-y||^2 + \lambda \ln \frac{f_{Y|X}(y|x)}{\int du f_{Y|X}(y|u) f_X(u)} \right) \end{aligned}$$

Optimization problem is: given f_X, find

$$\rho(\lambda) = \inf_{f_{Y|X}} \rho(\lambda, f_{Y|X}),$$

Rewrite the Lagrangian

$$\begin{aligned} &\rho(\lambda, f_{Y|X}) \\ &= \lambda \int dy f_Y(y) \int dx f_{X|Y}(x|y) \left(\frac{||x-y||^2}{\lambda} + \ln \frac{f_{X|Y}(x|y)}{f_X(x)} \right) \\ &= \lambda h(X) + \lambda \int dy f_Y(y) \int dx f_{X|Y}(x|y) \times \\ &\quad [-\ln e^{-\frac{||x-y||^2}{\lambda}} + \ln f_{X|Y}(x|y)] \end{aligned}$$

where

$$h(X) = \int f_X(x) \ln \frac{1}{f_X(x)} dx \qquad (30)$$

Shannon differential entropy.

Define

$$g_\lambda(x) \triangleq \frac{e^{-\frac{||x||^2}{2\sigma_g^2}}}{(2\pi\sigma_g^2)^{k/2}}$$

where $2\sigma_g^2 = \lambda$. Using the divergence inequality,

$$\begin{array}{l} \rho(\lambda, f_{Y|X}) \\ \quad = \lambda h(X) + \lambda \int dy f_Y(y) \int dx f_{X|Y}(x|y) \ln \frac{f_{X|Y}(x|y)}{g_\lambda(x-y)} - \frac{k\lambda}{2} \ln(2\pi\sigma_g^2) \\ \quad \geq \lambda h(X) - \frac{\lambda}{2} \ln(\pi\lambda)^k, \end{array}$$

achieved if $f_{X|Y}(x|y) = g_\lambda(x-y)$. The conditional density $f_{X|Y}$ is called the "backward channel distribution."

Suppose $f_{Y|X}$ is an arbitrary pdf yielding $I \leq R$ and distortion D. Then

$$D + \lambda I = \rho(\lambda, f_{Y|X}) \geq \lambda h(X) - \frac{\lambda}{2} \ln(\pi\lambda)^k \qquad (31)$$

and hence

$$\begin{aligned} D &\geq \lambda h(X) - \frac{\lambda}{2} \ln(\pi\lambda)^k - \lambda I \\ &\geq \lambda h(X) - \frac{\lambda}{2} \ln(\pi\lambda)^k - \lambda R \\ &= \lambda[h(X) - R - \frac{1}{2} \ln(\pi\lambda)^k] \end{aligned}$$

Bound holds for any value of λ, so maximize over λ.

Globally optimal value follows from the the $\ln r \leq r - 1$ inequality:

$$\begin{aligned} \lambda \frac{k}{2} \ln \frac{e^{\frac{2}{k}(h(X)-R)}}{\pi\lambda} &= \lambda \frac{k}{2} \left(\ln \frac{e^{\frac{2}{k}(h(X)-R)}}{\pi\lambda e} + 1 \right) \\ &\leq \lambda \frac{k}{2} \frac{e^{\frac{2}{k}(h(X)-R)}}{\pi\lambda e} = \frac{k e^{\frac{2}{k}(h(X)-R)}}{2\pi e} \end{aligned}$$

with equality iff

$$\lambda = \frac{e^{\frac{2}{k}(h(X)-R)}}{\pi e} \text{ erroneous } \lambda \text{ removed}$$

which yields the lower bound

$$D(R) \geq \frac{k}{2}\lambda == \frac{k}{2\pi e} e^{-\frac{2}{k}(R-h(X))} \triangleq D_{\text{SLB}}(R) \qquad (32)$$

Dual argument yields the Shannon lower bound to the RDF (what Shannon derived)

$$R(D) \geq h(X) - \frac{k}{2} \ln(\frac{2\pi e D}{k}) \triangleq R_{\text{SLB}}(D)$$

Recall the lower bound holds with equality iff $f_{X|Y}(x|y) = g_\lambda(x-y)$.

This will be true if one can find an f_Y for which

$$f_X(x) = \int g_\lambda(x-y) f_Y(y)\, dy,$$

which is not always possible.

Example where it is possible: 1 dimensional case with $f_X = \mathcal{N}(0,\sigma^2)$. Choosing $f_Y = \mathcal{N}(0,\sigma^2 - D)$ yields the Shannon lower bound with equality provided $R > 0$. $f_{X|Y}(x|y) = g_\lambda(x-y)$. Recall $\sigma_g^2 = \lambda/2$ and for equality in bound $\lambda = e^{2(h(X)-R)}/\pi e = 2D \Rightarrow \sigma_g^2 = D$. So X is $N(0,\sigma^2 - D + D = \sigma^2)$.

$$h(X) = \frac{1}{2}\ln(2\pi e\sigma^2)$$

$$R(D) = \frac{1}{2}\ln(2\pi e\sigma^2) - \frac{1}{2}\ln(2\pi eD) = \frac{1}{2}\ln\frac{\sigma^2}{D};\ D \in [0,\sigma^2]$$

and

$$D(R) = \sigma^2 2^{-2R};\ R \geq 0$$

It is known that the Shannon lower bound is tight as $R \to \infty$, that is, as $R \to \infty$, $D(R) - D_{\mathsf{SLB}}(R) \to 0$.

High-rate quantization theory

Bennett's approximations

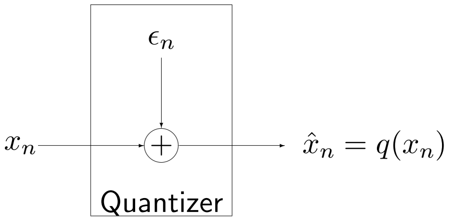
High rate, large N, large H, small distortion, fixed dimension

Suppose q is a simple scalar uniform quantizer with bin width Δ and N levels as in (10).

Define quantizer error $\epsilon = \epsilon(x) = q(x) - x$

Apply q to sequence X_n, $\epsilon_n = q(X_n) - X_n$

Sign chosen for "additive noise model" representation



"Model" because common in the literature to make assumptions on ϵ_n as behaving like signal-independent additive noise.

Consider system when Δ becomes small and M becomes large under the assumption that the input probability density function f_X is smooth.

First consider marginal cdf $F_{\epsilon_n}(\alpha) = \Pr(\epsilon_n \leq \alpha)$ and pdf $f_{\epsilon_n}(\alpha) = dF_{\epsilon_n}(\alpha)/d\alpha$, $\alpha \in (-\Delta/2, \Delta/2)$

$$\Pr(\epsilon_n \leq \alpha) = \sum_{k=0}^{N-1} \Pr(\epsilon_n \leq \alpha \text{ and } X_n \in S_k)$$

$$= \sum_{k=0}^{N-1} \Pr(a + (k + \frac{1}{2})\Delta - X \leq \alpha \text{ and } X \in [a + k\Delta, a + (k+1)\Delta))$$

Since the pdf smooth, mean value theorem $\Rightarrow$

$$\begin{aligned}\Pr(\epsilon_n \leq \alpha \text{ and } X_n \in S_k) &= \int_{a+(k+\frac{1}{2})\Delta-\alpha}^{a+(k+1)\Delta} f_{X_n}(\beta)\, d\beta \\ &\approx f_{X_n}(\mathcal{D}(k))(\alpha + \frac{\Delta}{2})\end{aligned}$$

so that

$$\Pr(\epsilon_n \le \alpha) \approx (\frac{\alpha}{\Delta} + \frac{1}{2}) \sum_{k=0}^{N-1} f_{X_n}(\mathcal{D}(k))\Delta$$

$$\approx (\frac{\alpha}{\Delta} + \frac{1}{2}) \int f_{X_n}(x)\, dx \approx (\frac{\alpha}{\Delta} + \frac{1}{2}), \alpha \in (-\frac{\Delta}{2}, \frac{\Delta}{2})$$

Riemann sum approximation to integral. ⇒

$$f_{\epsilon_n}(\alpha) \approx \frac{1}{\Delta} \text{ for } \alpha \in (-\frac{\Delta}{2}, \frac{\Delta}{2}),$$

consistent with the assumed behavior of the “additive noise” model.

In the high rate regime, average distortion of a uniform quantizer $\approx \Delta^2/12$ that predicted by adding uniform nose. Origin of “6dB per bit” improvement in SNR of a quantizer with a high bit rate.

Look at vectors $(\epsilon_n, \ldots, \epsilon_{n+k-1})$:

$$\Pr(\epsilon_{n+m} \le \alpha_m;\ m = 0, \ldots, k-1) =$$
$$\sum_{i_0,\ldots,i_{k-1}} \Pr(\epsilon_{n+m} \le \alpha_m \text{ and } X_{n+m} \in S_{i_m};\ m = 0, \ldots, k-1),$$

For $\alpha_m \in (-\Delta/2, \Delta/2)$, $m = 0, \ldots, k-1$

$$\Pr(\epsilon_{n+m} \le \alpha_m \text{ and } X_{n+m} \in S_{i_m};\ m = 0, \ldots, k-1) =$$
$$\int_{a+(k+\frac{1}{2})\Delta-\alpha_0}^{a+(k+1)\Delta} \cdots \int_{a+(k+\frac{1}{2})\Delta-\alpha_{k-1}}^{a+(k+1)\Delta} f_{X_n,\ldots,X_{n+k-1}}(\beta_0, \ldots, \beta_{k-1})\, d\beta_0 \cdots d\beta_{k-}$$
$$\approx f_{X_n,\ldots,X_{n+k-1}}(\mathcal{D}(i_0), \ldots, \mathcal{D}(i_{k-1}))(\alpha_0 + \frac{\Delta}{2}) \cdots (\alpha_{k-1} + \frac{\Delta}{2}) \quad (33)$$

whence

$$f_{\epsilon_n,\ldots,\epsilon_{n+k-1}}(\alpha_0, \ldots, \alpha_{k-1}) \approx \frac{1}{\Delta^k}$$

Thus in the high rate regime, the quantizer errors ≈ independent, hence white, and uniform!
But, requires very large N and very small cell size.

Eq (33) has an even stronger implication:

$$\Pr(\epsilon_m \le \alpha_m;\ m = 0, \ldots, k-1 | X_m \in S_{i_m};\ m = 0, \ldots, k-1) =$$
$$\frac{\Pr(\epsilon_m \le \alpha_m \text{ and } X_m \in S_{i_m};\ m = 0, \ldots, k-1)}{\Pr(X_m \in S_{i_m};\ m = 0, \ldots, k-1)} \approx$$
$$\frac{f_{X_n,\ldots,X_{n+k-1}}(\mathcal{D}(i_0), \ldots, \mathcal{D}(i_{k-1}))(\alpha_0 + \Delta/2) \cdots (\alpha_{k-1} + \Delta/2)}{f_{X_n,\ldots,X_{n+k-1}}(\mathcal{D}(i_0), \ldots, \mathcal{D}(i_{k-1}))\Delta^k}$$
$$= \frac{(\alpha_0 + \Delta/2) \cdots (\alpha_{k-1} + \Delta/2)}{\Delta^k}$$

$\Rightarrow f_{\epsilon_n,\ldots,\epsilon_{n+k-1}|X_n \in S_{i_0},\ldots,X_{n+k-1} \in S_{i_{k-1}}}(\alpha_0 \cdots \alpha_{k-1}) = 1/\Delta^k$
so that the errors are uniform and independent conditioned on each specific k-dimensional cell!.

These approximations have several implications:

- $E[\epsilon_n] \approx 0$, $\sigma^2_{\epsilon_n} \approx \Delta^2/12$.
- $R_\epsilon(n,m) \approx E[\epsilon_n \epsilon_m] \approx \sigma^2_\epsilon \delta_{n-m}$
- For $i = n, \ldots, n+k-1$

$$f_{\epsilon_i | X_m \in S_{i_m};\, m=0,\ldots,k-1}(\alpha) \approx \frac{1}{\Delta} \text{ for } \alpha \in (-\frac{\Delta}{2}, \frac{\Delta}{2})$$

  so that the marginal error is uniform conditioned on k quantizer outputs including the same index. This leads to

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$$f_{\epsilon_m | X_n \in S_\ell}(\alpha) \approx \frac{1}{\Delta} \text{ for } \alpha \in (-\frac{\Delta}{2}, \frac{\Delta}{2}) \Rightarrow E[\epsilon_m | q(X_n) = \mathcal{D}(\ell)] \approx 0$$

- Since $q(X_n) = \mathcal{D}(l)$ iff $X_n \in S_l$,

$$\begin{aligned} R_{q,\epsilon}(n,m) &= E[q(X_n)\epsilon_m] \\ &= \sum_{l=0}^{M-1} \mathcal{D}(l) E[\epsilon_m | q(X_n) = \mathcal{D}(\ell)] \Pr(q(X_n) = \mathcal{D}(\ell)) = 0 \end{aligned}$$

  so that the quantizer output is uncorrelated with the input.

- The previous formula implies that

$$\begin{aligned} R_{X,\epsilon}(n,m) &= E[X_n \epsilon_m] \\ &= E[(q(X_n) - \epsilon_n)\epsilon_m] \\ &= E[q(X_n)\epsilon_m] - R_\epsilon(n,m) \\ &= -R_\epsilon(n,m), \end{aligned}$$

i.e., that

$$R_{X,\epsilon}(n,m) = -\sigma^2_\epsilon \delta_{n-m}.$$

input and quantizer error are *not uncorrelated!*

Hence common assumption of independence between the quantizer error and the input will yield incorrect results.

Bennett extended his approximations to nonuniform scalar quantizers and used his approximations to quantify the optimal performance in fixed-rate systems. His methods, however, do not extend to the multidimensional case so we shall proceed to more general results for vectors.