EE 372 Quantization

18 April 2007 Spring 2006-2007

Homework #2

Due: 2 May 2007

1. A *random quantizer* q is defined by a partition $\{S_i\}$ of the real line, a set of output points $\mathcal{D}(j)$ (for $j = 1, 2, \cdots, N$) with $\mathcal{D}(j)$ contained in S_j, and a set of conditional probabilities $q(j|i)$ such that
$$q(j|i) = P[Y = \mathcal{D}(j)|X \in S_i].$$
For a given partition, and a given set of output points determine the optimal set of conditional probabilities to minimize mean square quantization error.

2. Weighted quadratic distortion measures

   Suppose that a distortion measure on $\Re^k$ is defined by
$$d(x, y) = (x - y)^t W_x (x - y), \tag{1}$$
where W_x is a (strictly) positive definite matrix for all x. In the special case where $W_x = W$ and W is an inverse covariance matrix, this is called the *Mahalonobis distance*.

   Show that if you have a probability distribution P_X in the input vector, then the Lloyd centroid of a set S is given by
$$\text{cent}(S) = E[W_X|X \in S]^{-1} E[W_X X|X \in S]$$
assuming that all the given expectations are finite.

   Show also the following analogs to the properties of centroidal quantizers for squared error distortion:
$$\begin{aligned} E(W_X X) &= E(W_X q(X)) & (2) \\ D(q) &= E(X^t W_X X) - E(q(X)^t W_X q(X)) & (3) \end{aligned}$$

3. *The partial distortion theorem* Use Gersho's conjectures and approximations to argue that for fixed rate vector quantization with very large N, the partial distortions
$$D_i = \int_{S_i} ||x - \mathcal{D}(i)||^2 \, dP_X(x)$$
are approximately constant for all i and describe the constant. In contrast, use Gersho's arguments to show that for the variable rate case it is the volumes
$$V(S_i) = \int_{S_i} dx$$
that are approximately constant. (For the mathematically inclined, the volume is the Lebesgue measure of the set.)

4. U is a uniform random variable on $[-\frac{1}{2}, \frac{1}{2}]$, i.e., it has probability density function (pdf)

$$f_U(u) = \begin{cases} 1 & if -\frac{1}{2} \leq u \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

W is a discrete random variable with probability mass function (pmf)

$$p_W(w) = \begin{cases} \frac{1}{2} & if w = 1 \\ \frac{1}{2} & if w = 2 \end{cases}$$

The two random variables U and W are mutually independent.

The random variable X is defined by $X = WU$.

(a) Find and sketch the pdf for X. (This is just to make sure you do this carefully as you will need this pdf.)

(b) Describe quantizers q_N with $N = 2$ and $N = 4$ levels for X that satisfy the Lloyd conditions and evaluate the average distortion $E[|X - q_N(X)|^2]$.

(c) Let q_N be a fixed-rate scalar quantizer for X for large N that is optimal in the sense of minimizing the average distortion. Find an approximation for $E[|X - q_N(X)|^2]$ and describe the optimal quantizer, i.e., give the quantizer point density and describe in words a quantizer having this point density.

(d) Repeat the previous part for a variable-rate quantizer.

(e) Find the Shannon lower bound to the distortion rate function, $D_{\text{SLB}}(R)$, for X with respect to a squared error distortion measure.

5. *Differential entropy of a Gaussian random vector*

Suppose that X is a k-dimensional zero mean Gaussian random vector with pdf

$$f_X(x) = \frac{e^{-\frac{1}{2}x' R^{-1} x}}{(2\pi)^{n/2} \text{det}(R)^{1/2}},$$

where R is the $n \times n$ covariance matrix with entries r_{k-j}.

(a) Show by direct multidimensional integration using the properties of a Gaussian vector that X has differential entropy

$$h(f_X) = -\int f_X(x) \ln f_X(x)\, dx = \frac{1}{2} \ln(2\pi e)^n \text{det} R.$$

(b) Show that if g is any pdf with 0 mean and covariance $R = E_g(XX^t)$, then $h(g) \leq h(f_X)$ with equality if and only if $g - f_X$. In words, a Gaussian random vector has the largest differential entropy over all random vectors with the same coveriance.
Hint: Use the divergence inequality and the matrix equality

$$x^t R^{-1} x = \text{Tr}(R^{-1} x x^t)$$

where Tr denotes the trace of a matrix, the sum of its diagonal elements.

(c) Write the high-rate approximations to the operational distortion-rate and rate-distortion functions for f_X for both the fixed rate and variable rate cases.

(d) Show that the Gaussian is the worst case source for variable rate quantization in the high rate regime for a given covariance, i.e., that the operational rate-distortion function or distortion-rate function is the largest given the vector covariance if the vector is Gaussian.

(e) Find the Shannon lower bound to the distortion rate function for the Gaussian random vector X and compare it to the high rate approximations.

6. Evaluate the high rate quantization approximations to the operational distortion-rate function for both fixed rate and variable rate (entropy coding) for a k dimensional random vector $X = (X_0, X_1, \ldots, X_{k-1})$ formed as the first k samples of a random process $\{X_n\}$ described below. Compare the result to the Shannon lower bound to the Shannon distortion-rate function. Also show the rate-distortion forms.

   - $\{X_n\}$ is an iid Gaussian random process with 0 mean and variance σ_X.
   - $\{X_n\}$ is a Gauss Markov random process with zero mean, correlation coefficient r and variance σ^2. (i.e., the autocorrelation function of X_n is $R_X(k,n) = \sigma^2 r^{|k-n|}$), where $|r| < 1$.

   For the fixed rate case, what is the optimal Gersho quantizer point density function? What can you say about the point density function for the variable rate case?

   What can you say about the limits of the above quantities if $k \to \infty$?