EE364a Review Session 4

session outline:

- transformations
- Schur complements

Transformations

- transformation of objective
- transformation of constraints

example: objective transformation

$$
\begin{array}{ll}
\text{minimize} & \int_{-\infty}^{c^T x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\
\text{subject to} & Ax \preceq b
\end{array}
$$

- is it a convex problem?
- is it a quasiconvex problem?

solution:

- nonconvex: objective is not convex
- quasilinear: sublevel and superlevel sets are convex
- $\int_{-\infty}^{c^T x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt = \Phi(c^T x)$ where $\Phi(u)$ is monotone increasing in u, so minimizing $\Phi(c^T x)$ is the same as minimizing $c^T x$
- thus equivalent problem is an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

example: transform the following constraint to a set of linear constraints

$$a_1^T x + b_1 + \max(a_2^T x + b_2, a_3^T x + b_3) \leq 0$$

solution 1:

- introduce a new variable t

- thus

$$\begin{array}{rcl} a_1^T x + b_1 + t & \leq & 0 \\ a_2^T x + b_2 - t & \leq & 0 \\ a_3^T x + b_3 - t & \leq & 0 \end{array}$$

solution 2:

- put $a_1^T x + b_1$ inside the $\max$ function

- then we get

$$\begin{array}{rcl} (a_1 + a_2)^T x + (b_1 + b_2) & \leq & 0 \\ (a_1 + a_3)^T x + (b_1 + b_3) & \leq & 0 \end{array}$$

example: what about the following constraint?

$$a_1^T x + b_1 - \max(a_2^T x + b_2, a_3^T x + b_3) \leq 0$$

solution

- non-convex constraint
- cannot be transformed into a set of linear inequalities
- consider the following problem

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_1^T x + b_1 - \max(a_2^T x + b_2, a_3^T x + b_3) \leq 0 \end{array}$$

This is not an LP, but can be solved easily by solving two LPs.

Consider the last inequality:

– if $a_2^T x + b_2 \geq a_3^T x + b_3$, then $a_1^T x + b_1 - a_2^T x - b_2 \leq 0$
– if $a_2^T x + b_2 \leq a_3^T x + b_3$, then $a_1^T x + b_1 - a_3^T x - b_3 \leq 0$

Thus, optimal solution can be found by solving two LPs:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_2^T x + b_2 \geq a_3^T x + b_3 \\ & a_1^T x + b_1 - a_2^T x - b_2 \leq 0 \end{array}$$

and

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_2^T x + b_2 \leq a_3^T x + b_3 \\ & a_1^T x + b_1 - a_3^T x - b_3 \leq 0 \end{array}$$

Then choose optimal solution with smaller objective value.

example: problem 4.58

$$\begin{array}{ll} \text{maximize} & \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{subject to} & k_{t+1} = k_t + f(k_t) - c_t, \quad t = 0, \ldots, T-1 \\ & k_t \geq 0, \quad t = 0, \ldots, T \end{array}$$

$u(c_t)$ is concave, increasing in c_t

solution

- equality constraint is not convex
- can 'convexify' by loosening to inequality

$$\begin{array}{ll} \text{maximize} & \sum_{t=0}^{T} \beta^t u(c_t) \\ \text{subject to} & k_{t+1} \leq k_t + f(k_t) - c_t, \quad t = 0, \ldots, T-1 \\ & k_t \geq 0, \quad t = 0, \ldots, T \end{array}$$

- at optimal $k^\star$, $c^\star$ equality always holds

- if, for some ‘optimal’ $\tilde{k}$, $\tilde{c}$ at time t:

$$\tilde{k}_{t+1} < \tilde{k}_t + f(\tilde{k}_t) - \tilde{c}_t$$

  - we can increase $\tilde{c}_t$ until equality is achieved
  - objective increases
  - therefore $\tilde{k}$, $\tilde{c}$ was not optimal

example:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & h(x) = 0 \end{array}$$

- when f_0, f_i are convex, h is affine, this is standard form

- when h is convex, when can we write:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & h(x) \leq 0 \end{array}$$

solution

- we want $h(x^\star) = 0$

- suppose $h(x) < 0$: we want to increase $h(x)$

- suppose $h(x)$ is monotonically decreasing in x_r
  - we can decrease x_r, increasing $h(x)$ to equality
  - $f_0(x)$ improves if it is monotonically increasing in x_r
  - $f_1(x), \ldots f_m(x)$ must be nondecreasing in x_r

- suppose $h(x)$ is monotonically increasing in x_r
  - we can increase x_r, increasing $h(x)$ to equality
  - $f_0(x)$ improves if it is monotonically decreasing in x_r
  - $f_1(x), \ldots, f_m(x)$ must be nonincreasing in x_r

Schur complements

Let $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in \mathbf{S}^n$

- $X \succ 0 \iff A \succ 0$ and $S = C - B^T A^{-1} B \succ 0$

- minimizing over one component preserves convexity:

$$\inf_u \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = v^T S v$$

- if $A \succ 0$ then $X \succeq 0 \iff S \succeq 0$

example: convert $\|Ax - b\|_2 \leq \gamma, \gamma > 0$ to SDP form

solution

- can square both sides

$$(Ax - b)^T (Ax - b) \leq \gamma^2$$

- using Schur complement trick, can write as:

$$\begin{bmatrix} I & Ax - b \\ (Ax - b)^T & \gamma^2 \end{bmatrix} \succeq 0$$