Convex optimization examples

- multi-period processor speed scheduling
- minimum time optimal control
- grasp force optimization
- optimal broadcast transmitter power allocation
- phased-array antenna beamforming
- optimal receiver location

Multi-period processor speed scheduling

- processor adjusts its speed $s_t \in [s^{\min}, s^{\max}]$ in each of T time periods
- energy consumed in period t is $\phi(s_t)$; total energy is $E = \sum_{t=1}^{T} \phi(s_t)$
- n jobs
  - job i available at $t = A_i$; must finish by deadline $t = D_i$
  - job i requires total work $W_i \geq 0$
- $\theta_{ti} \geq 0$ is fraction of processor effort allocated to job i in period t

$$\mathbf{1}^T \theta_t = 1, \qquad \sum_{t=A_i}^{D_i} \theta_{ti} s_t \geq W_i$$

- choose speeds s_t and allocations θ_{ti} to minimize total energy E

Minimum energy processor speed scheduling

- work with variables $S_{ti} = \theta_{ti} s_t$

$$s_t = \sum_{i=1}^{n} S_{ti}, \qquad \sum_{t=A_i}^{D_i} S_{ti} \geq W_i$$

- solve convex problem

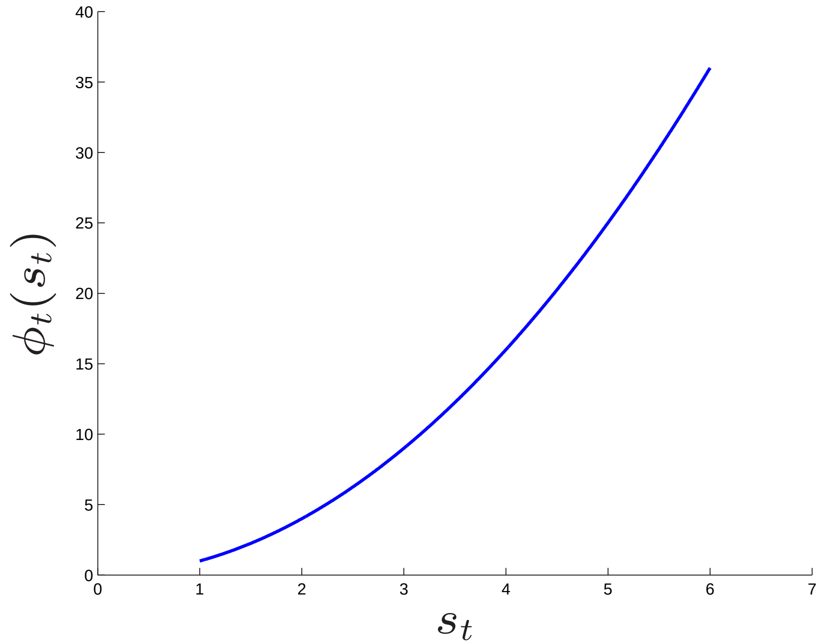
$$\begin{array}{ll} \text{minimize} & E = \sum_{t=1}^{T} \phi(s_t) \\ \text{subject to} & s^{\min} \leq s_t \leq s^{\max}, \quad t = 1, \ldots, T \\ & s_t = \sum_{i=1}^{n} S_{ti}, \quad t = 1, \ldots, T \\ & \sum_{t=A_i}^{D_i} S_{ti} \geq W_i, \quad i = 1, \ldots, n \end{array}$$

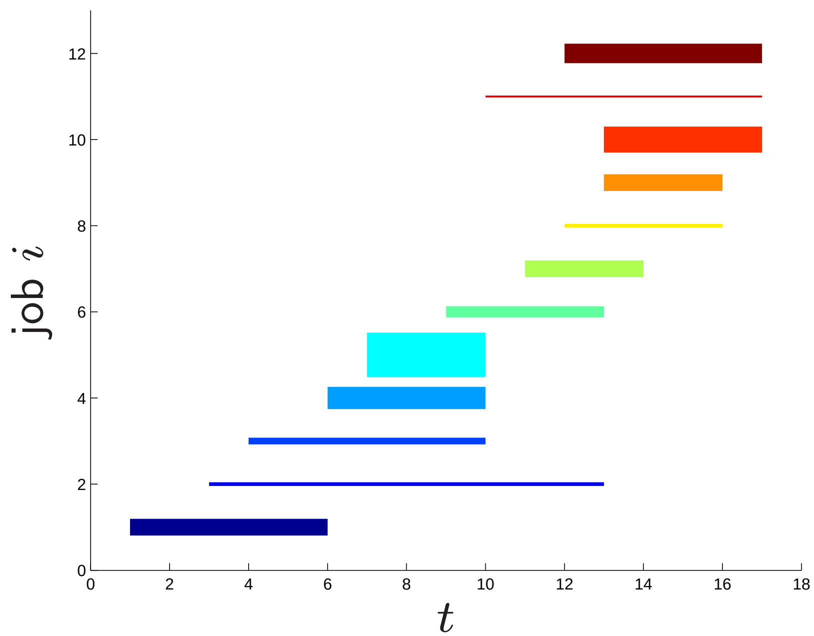
- a convex problem when ϕ is convex

- can recover $\theta_t^\star$ as $\theta_{ti}^\star = (1/s_t^\star) S_{ti}^\star$

Example

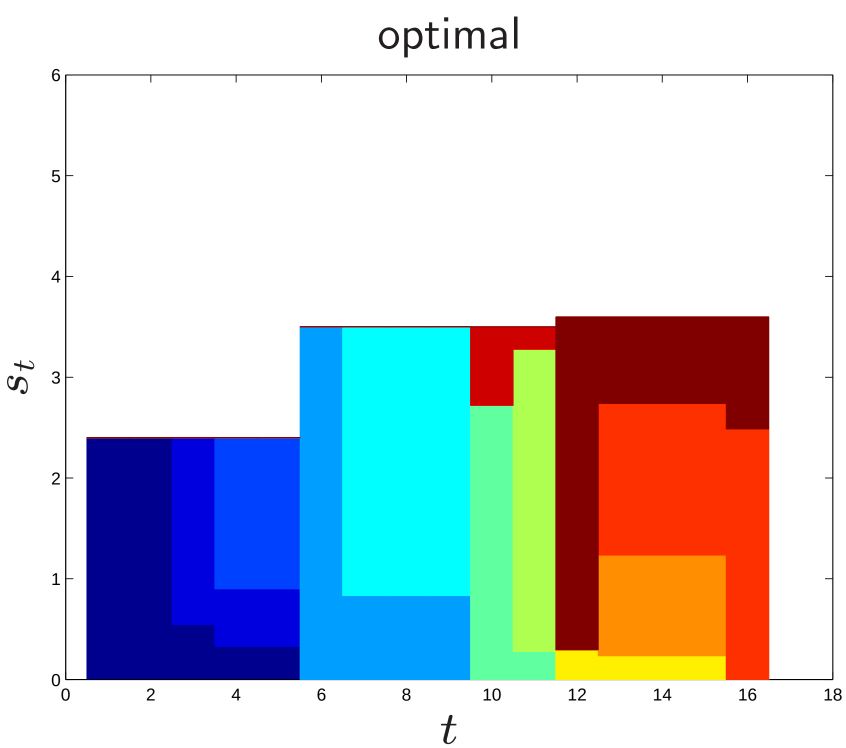
- $T = 16$ periods, $n = 12$ jobs
- $s^{\min} = 1$, $s^{\max} = 6$, $\phi(s_t) = s_t^2$
- jobs shown as bars over $[A_i, D_i]$ with area $\propto W_i$

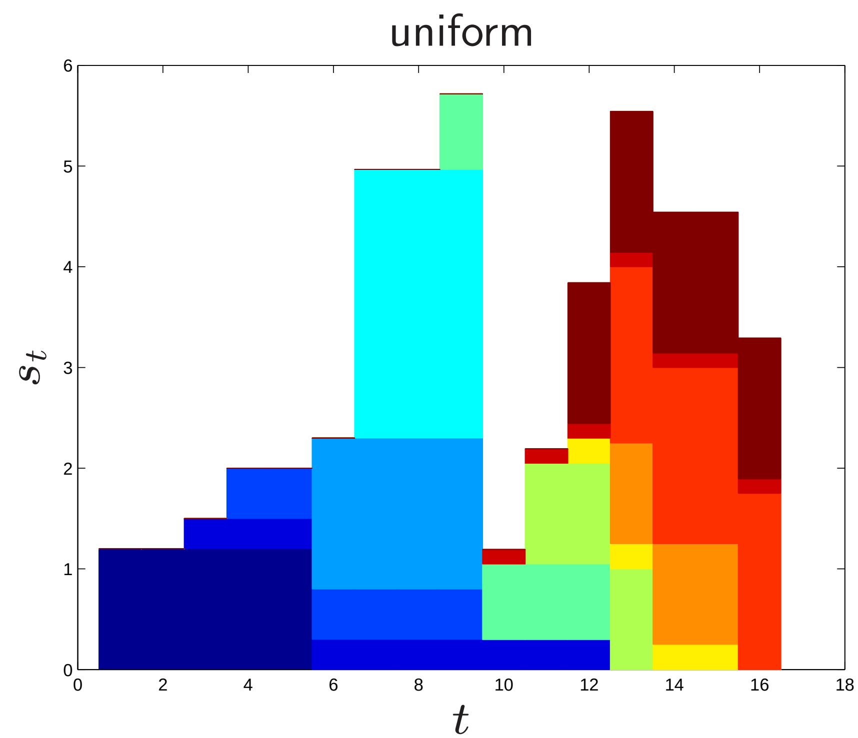




Optimal and uniform schedules

- uniform schedule: $S_{ti} = W_i/(D_i - A_i + 1)$; gives $E^{\mathrm{unif}} = 204.3$
- optimal schedule: $S^\star_{ti}$; gives $E^\star = 167.1$





Minimum-time optimal control

- linear dynamical system:

$$x_{t+1} = Ax_t + Bu_t, \quad t = 0, 1, \ldots, K, \qquad x_0 = x^{\text{init}}$$

- inputs constraints:

$$u_{\min} \preceq u_t \preceq u_{\max}, \quad t = 0, 1, \ldots, K$$

- minimum time to reach state x_{des}:

$$f(u_0, \ldots, u_K) = \min \{T \mid x_t = x_{\text{des}} \text{ for } T \leq t \leq K+1\}$$

state transfer time f is quasiconvex function of $(u_0, \ldots, u_K)$:

$$f(u_0, u_1, \ldots, u_K) \leq T$$

if and only if for all $t = T, \ldots, K+1$

$$x_t = A^t x^{\text{init}} + A^{t-1} B u_0 + \cdots + B u_{t-1} = x_{\text{des}}$$

i.e., sublevel sets are affine

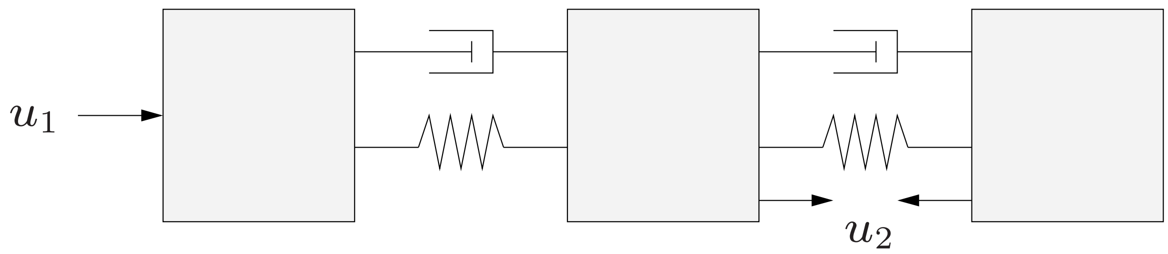
minimum-time optimal control problem:

$$\begin{array}{ll} \text{minimize} & f(u_0, u_1, \ldots, u_K) \\ \text{subject to} & u_{\min} \preceq u_t \preceq u_{\max}, \quad t = 0, \ldots, K \end{array}$$

with variables $u_0, \ldots, u_K$

a quasiconvex problem; can be solved via bisection

Minimum-time control example

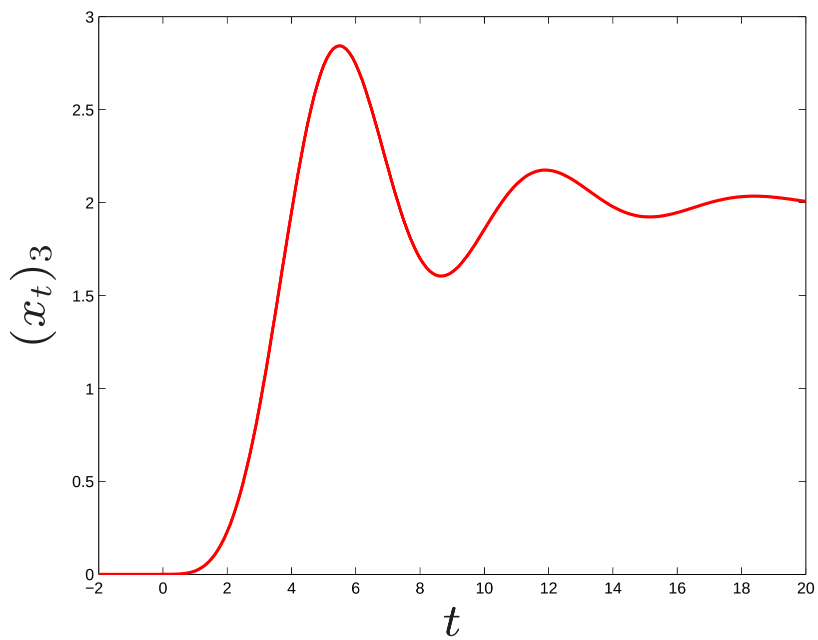


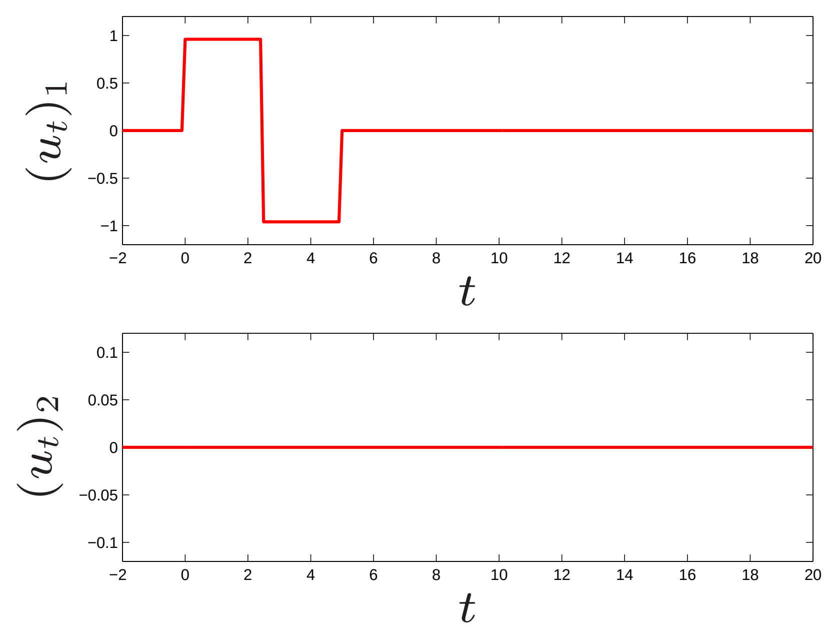
- force $(u_t)_1$ moves object modeled as 3 masses (2 vibration modes)
- force $(u_t)_2$ used for active vibration suppression
- goal: move object to commanded position as quickly as possible, with

$$|(u_t)_1| \leq 1, \qquad |(u_t)_2| \leq 0.1, \qquad t = 0, \ldots, K$$

Ignoring vibration modes

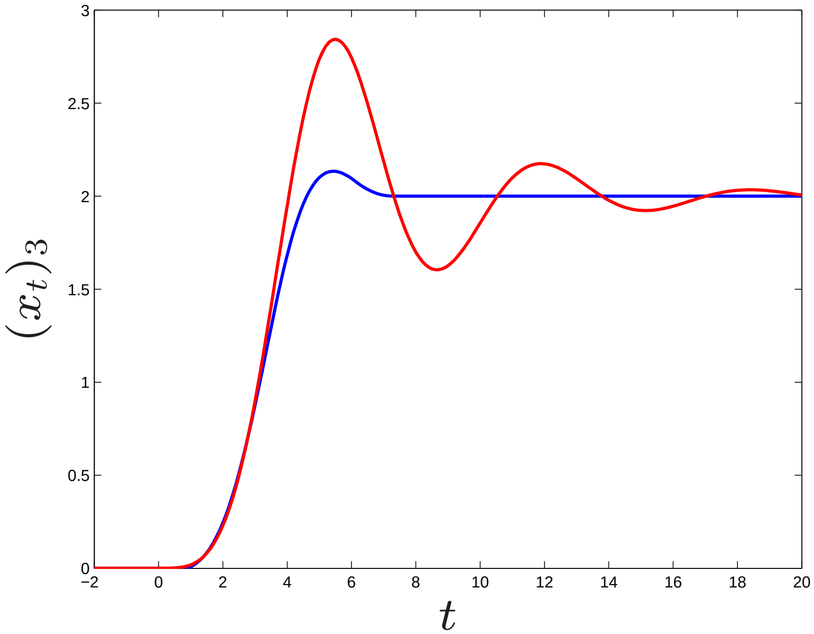
- treat object as single mass; apply only u_1
- analytical ('bang-bang') solution

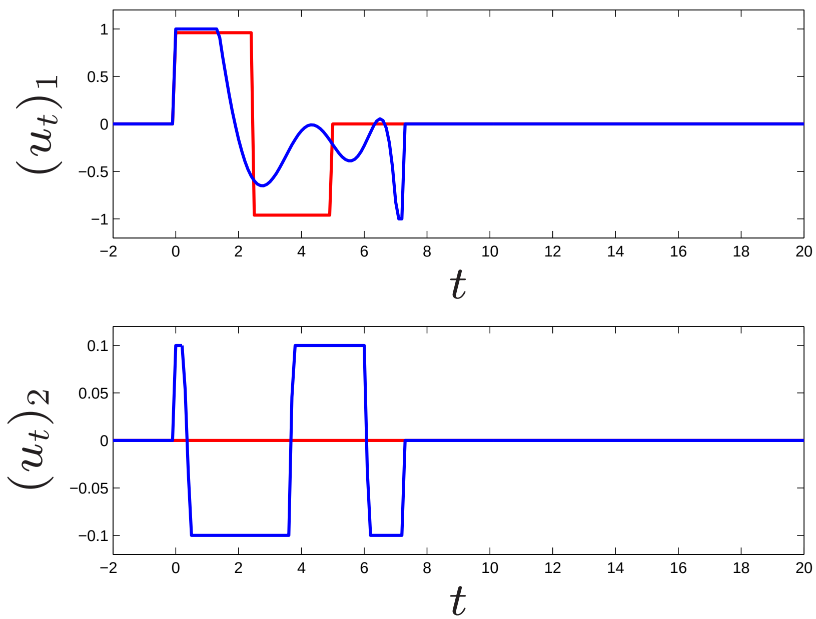




With vibration modes

- no analytical solution
- a quasiconvex problem; solved using bisection





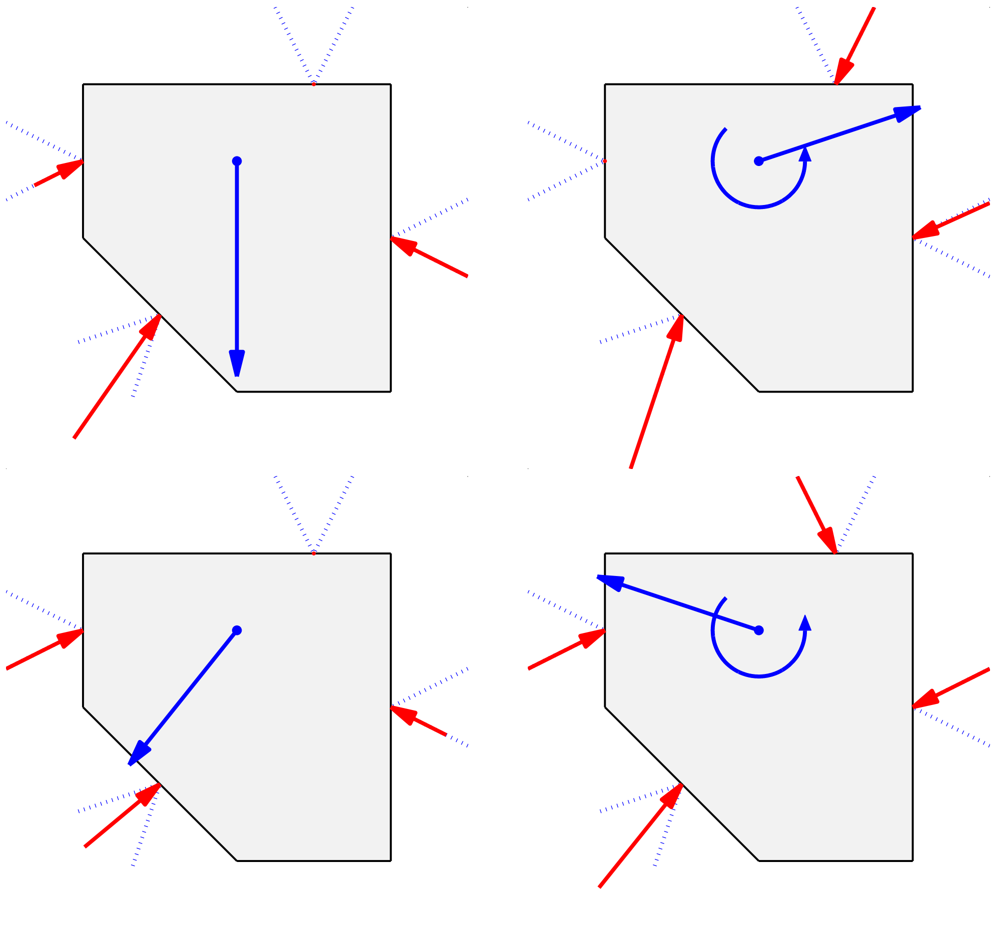
Grasp force optimization

- choose K grasping forces on object
  - resist external wrench
  - respect friction cone constraints
  - minimize maximum grasp force
- convex problem (second-order cone program):

$$
\begin{array}{lll}
\text{minimize} & \max_i \|f^{(i)}\|_2 & \textit{max contact force} \\
\text{subject to} & \sum_i Q^{(i)} f^{(i)} = f^{\text{ext}} & \textit{force equillibrium} \\
& \sum_i p^{(i)} \times (Q^{(i)} f^{(i)}) = \tau^{\text{ext}} & \textit{torque equillibrium} \\
& \mu_i f_3^{(i)} \geq \left(f_1^{(i)2} + f_2^{(i)2} \right)^{1/2} & \textit{friction cone contraints}
\end{array}
$$

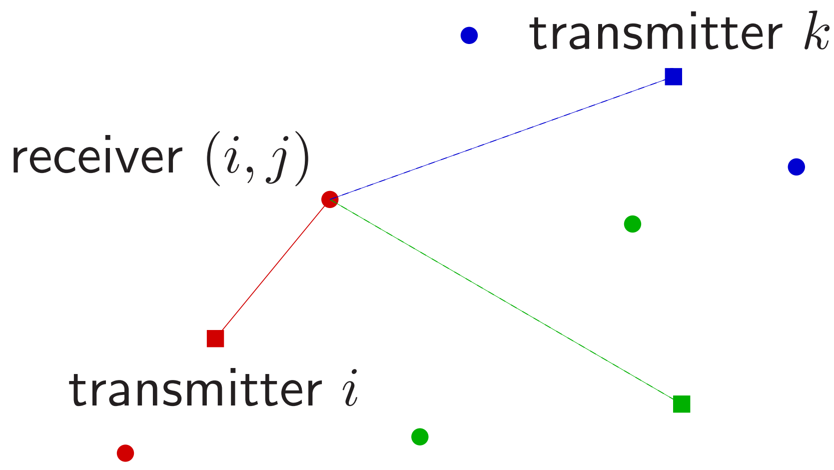
variables $f^{(i)} \in \mathbf{R}^3$, $i = 1, \ldots, K$ (contact forces)

Example



Optimal broadcast transmitter power allocation

- m transmitters, mn receivers all at same frequency
- transmitter i wants to transmit to n receivers labeled (i, j), $j = 1, \ldots, n$
- A_{ijk} is path gain from transmitter k to receiver (i, j)
- N_{ij} is (self) noise power of receiver (i, j)
- variables: transmitter powers p_k, $k = 1, \ldots, m$



at receiver (i, j):

- signal power:

$$S_{ij} = A_{iji} p_i$$

- noise plus interference power:

$$I_{ij} = \sum_{k \neq i} A_{ijk} p_k + N_{ij}$$

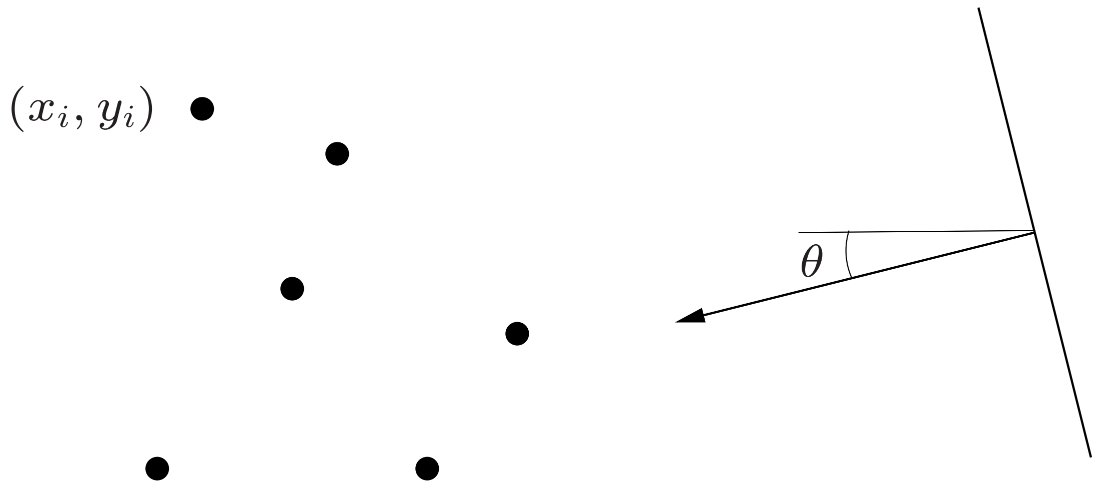
- signal to interference/noise ratio (SINR): S_{ij}/I_{ij}

problem: choose p_i to maximize smallest SINR:

$$\begin{array}{ll} \text{maximize} & \displaystyle \min_{i,j} \frac{A_{iji} p_i}{\sum_{k \neq i} A_{ijk} p_k + N_{ij}} \\ \text{subject to} & 0 \leq p_i \leq p_{\mathsf{max}} \end{array}$$

. . . a (generalized) linear fractional program

Phased-array antenna beamforming



- omnidirectional antenna elements at positions (x_1, y_1), . . . , (x_n, y_n)
- unit plane wave incident from angle θ induces in ith element a signal $e^{j(x_i \cos\theta + y_i \sin\theta - \omega t)}$

  ($j = \sqrt{-1}$, frequency ω, wavelength 2π)

- demodulate to get output $e^{j(x_i \cos\theta + y_i \sin\theta)} \in \mathbf{C}$

- linearly combine with complex weights w_i:

$$y(\theta) = \sum_{i=1}^{n} w_i e^{j(x_i \cos\theta + y_i \sin\theta)}$$

- $y(\theta)$ is (complex) *antenna array gain pattern*

- $|y(\theta)|$ gives sensitivity of array as function of incident angle θ

- depends on design variables $\mathbf{Re}\, w$, $\mathbf{Im}\, w$
  (called *antenna array weights* or *shading coefficients*)

design problem: choose w to achieve desired gain pattern

Sidelobe level minimization

make $|y(\theta)|$ small for $|\theta - \theta_{\mathrm{tar}}| > \alpha$

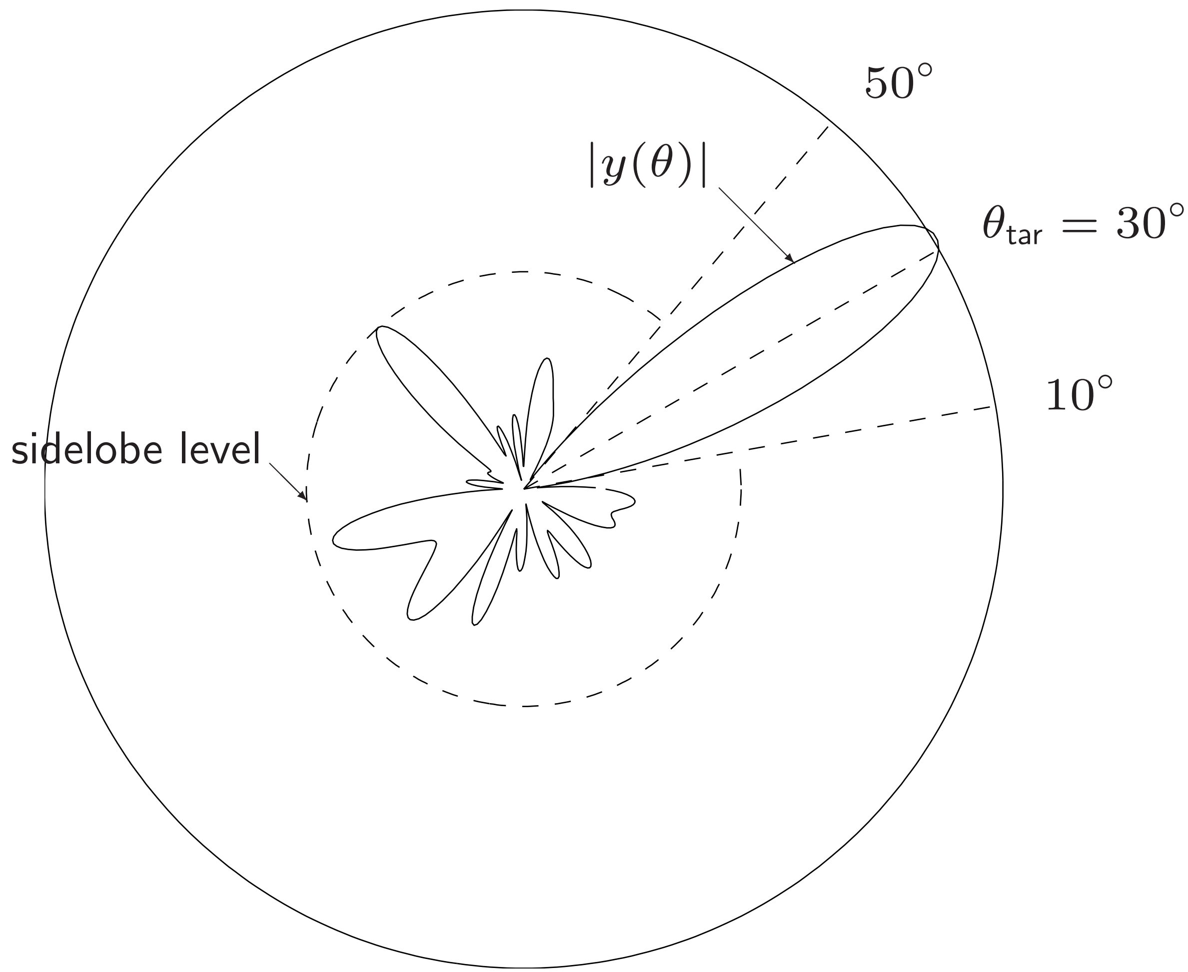
(θ_{tar}: target direction; 2α: beamwidth)

via least-squares (discretize angles)

$$\begin{array}{ll} \text{minimize} & \sum_i |y(\theta_i)|^2 \\ \text{subject to} & y(\theta_{\mathrm{tar}}) = 1 \end{array}$$

(sum is over angles outside beam)

least-squares problem with two (real) linear equality constraints



50°
$|y(\theta)|$
$\theta_{\text{tar}} = 30°$
10°
sidelobe level

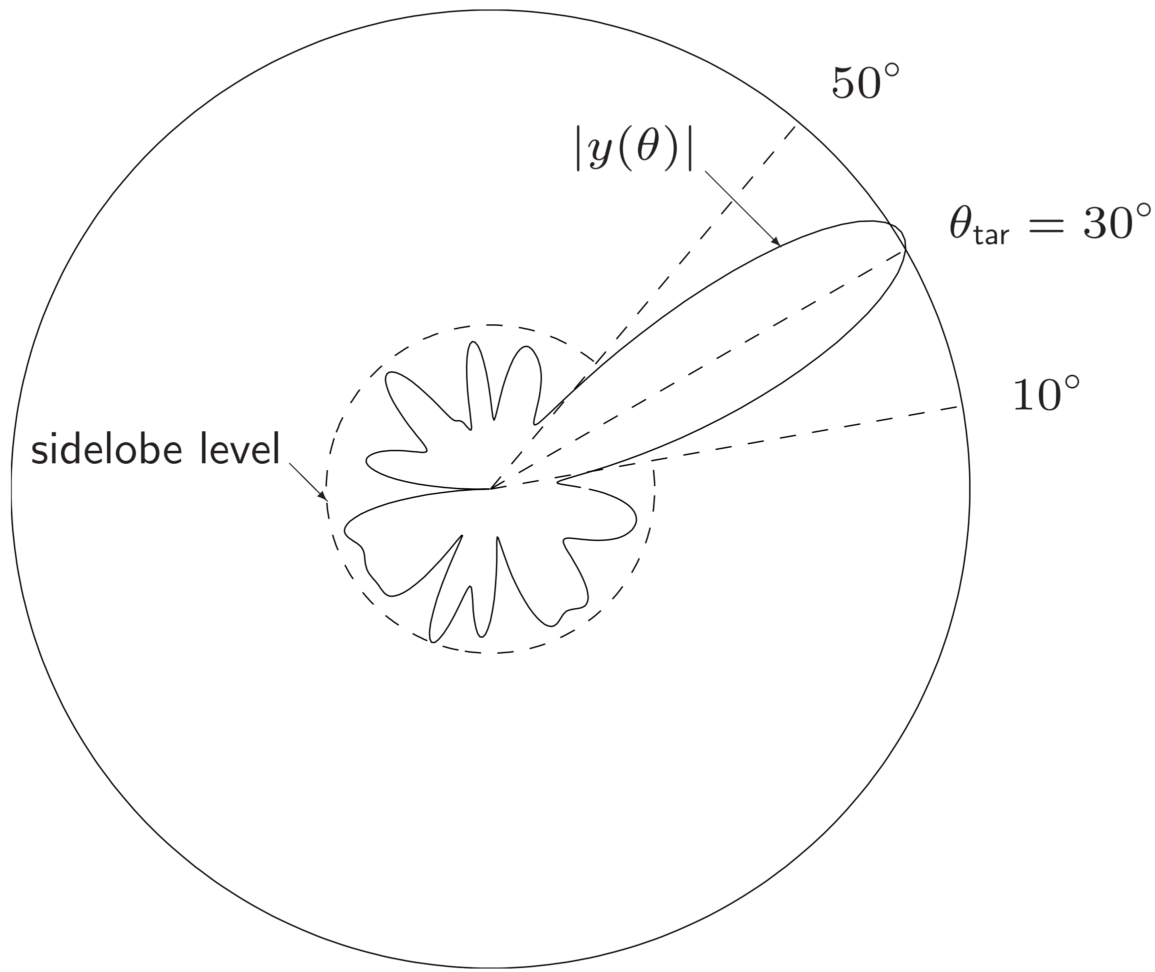
minimize sidelobe level (discretize angles)

$$\begin{array}{ll} \text{minimize} & \max_i |y(\theta_i)| \\ \text{subject to} & y(\theta_{\text{tar}}) = 1 \end{array}$$

(max over angles outside beam)

can be cast as SOCP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & |y(\theta_i)| \leq t \\ & y(\theta_{\text{tar}}) = 1 \end{array}$$


50°
$|y(\theta)|$
$\theta_{\text{tar}} = 30°$
10°
sidelobe level

Extensions

convex (& quasiconvex) extensions:

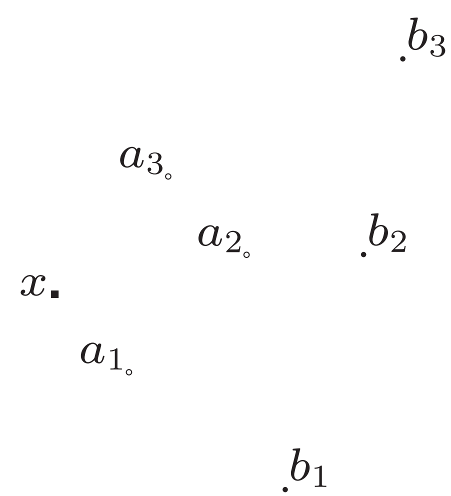
- $y(\theta_0) = 0$ (null in direction θ_0)
- w is real (amplitude only shading)
- $|w_i| \leq 1$ (attenuation only shading)
- minimize $\sigma^2 \sum_{i=1}^{n} |w_i|^2$ (thermal noise power in y)
- minimize beamwidth given a maximum sidelobe level

nonconvex extension:

- maximize number of zero weights

Optimal receiver location

- N transmitter frequencies $1, \ldots, N$
- transmitters at locations $a_i,\ b_i \in \mathbf{R}^2$ use frequency i
- transmitters at a_1, a_2, . . . , a_N are the wanted ones
- transmitters at b_1, b_2, . . . , b_N are interfering
- receiver at position $x \in \mathbf{R}^2$



- (signal) receiver power from a_i: $\|x - a_i\|_2^{-\alpha}$ ($\alpha \approx 2.1$)

- (interfering) receiver power from b_i: $\|x - b_i\|_2^{-\alpha}$ ($\alpha \approx 2.1$)

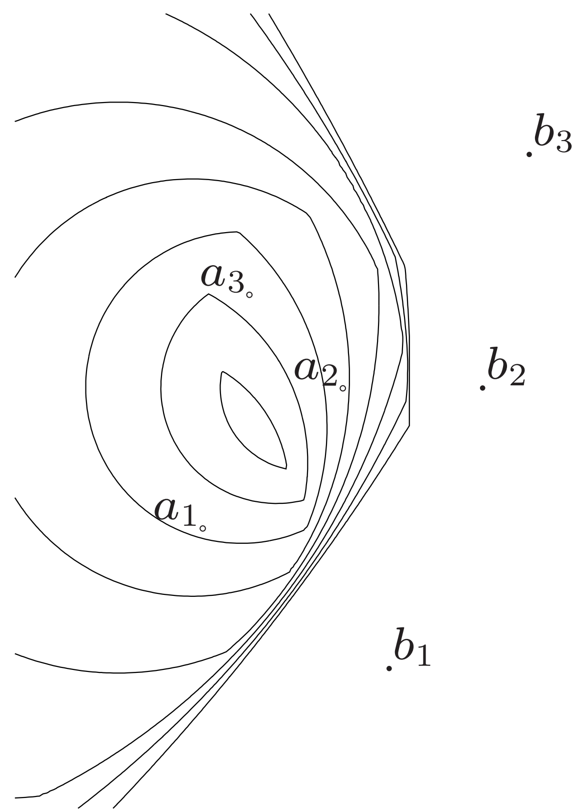
- worst signal to interference ratio, over all frequencies, is

$$\mathsf{S/I} = \min_i \frac{\|x - a_i\|_2^{-\alpha}}{\|x - b_i\|_2^{-\alpha}}$$

- what receiver location x maximizes S/I?

S/I is quasiconcave on $\{x \mid \text{S/I} \geq 1\}$, *i.e.*, on

$$\{x \mid \|x - a_i\|_2 \leq \|x - b_i\|_2,\ i = 1, \ldots, N\}$$



can use bisection; every iteration is a convex quadratic feasibility problem