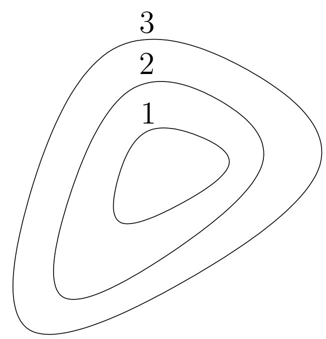
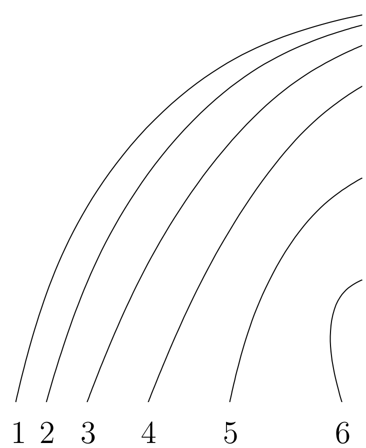
EE364a Homework 2 solutions

3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.

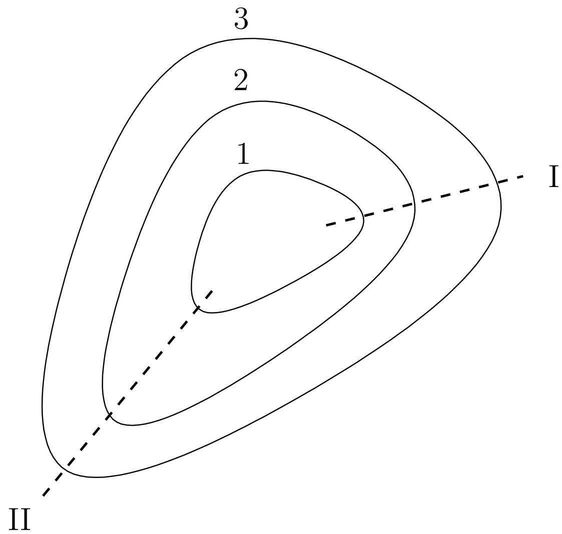


Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.

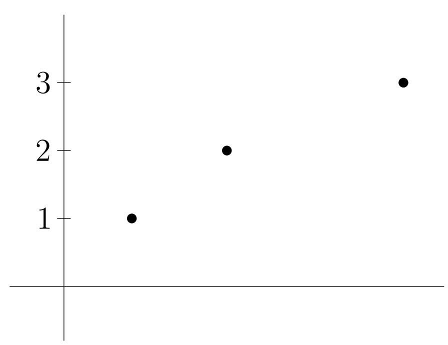


Solution. The first function could be quasiconvex because the sublevel sets appear to be convex. It is definitely not concave or quasiconcave because the superlevel sets are not convex.

It is also not convex, for the following reason. We plot the function values along the dashed line labeled I.



Along this line the function passes through the points marked as black dots in the figure below. Clearly along this line segment, the function is not convex.



If we repeat the same analysis for the second function, we see that it could be concave (and therefore it could be quasiconcave). It cannot be convex or quasiconvex, because the sublevel sets are not convex.

3.3 *Inverse of an increasing convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, *i.e.*, the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

Solution. g is concave. Its hypograph is

$$\begin{aligned}
\mathbf{hypo}\, g &= \{(y,t) \mid t \leq g(y)\} \\
&= \{(y,t) \mid f(t) \leq f(g(y))\} \quad \text{(because } f \text{ is increasing)} \\
&= \{(y,t) \mid f(t) \leq y\} \\
&= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{epi}\, f.
\end{aligned}$$

The concavity of g can also be proven using Jensen's inequality. Consider any two points y_1 and y_2 in $\mathbf{dom}\, g$, and let $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Then

$$\begin{aligned}
\theta g(y_1) + (1-\theta) g(y_2) &= \theta x_1 + (1-\theta) x_2 \\
&= g(f(\theta x_1 + (1-\theta)x_2)) \quad \text{(by definition of inverse)} \\
&\leq g(\theta f(x_1) + (1-\theta) f(x_2)) \\
&= g(\theta y_1 + (1-\theta) y_2).
\end{aligned}$$

Because f is increasing, then its inverse g is also increasing. The inequality step follows from the convexity of f because $f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$, which implies that $g(f(\theta x_1 + (1-\theta)x_2)) \leq g(\theta f(x_1) + (1-\theta) f(x_2))$.

For differentiable g, f, we can also prove the result as follows. Differentiate $g(f(x)) = x$ once to get

$$g'(f(x)) = 1/f'(x).$$

so g is increasing. Differentiate again to get

$$g''(f(x)) = -\frac{f''(x)}{f'(x)^3},$$

so g is concave.

3.6 *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

Solution. The epigraph of f is a halfspace if and only if f is affine.

The epigraph of f is a convex cone if and only if f is convex and positively homogeneous, *i.e.*, $f(\alpha x) = \alpha f(x)$ for any x and any $\alpha \geq 0$.

The epigraph of f is a polyhedron if and only if f is convex and piecewise affine.

3.15 *A family of concave utility functions.* For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\, u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\, u_0 = \mathbf{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation.*

Solution.

(a) In this limit, both the numerator and denominator go to zero, so we use l'Hopital's rule:

$$\lim_{\alpha\to 0} u_\alpha(x) = \lim_{\alpha\to 0} \frac{(d/d\alpha)(x^\alpha - 1)}{(d/d\alpha)\alpha} = \lim_{\alpha\to 0} \frac{x^\alpha \log x}{1} = \log x.$$

(b) By inspection we have

$$u_\alpha(1) = \frac{1^\alpha - 1}{\alpha} = 0.$$

The derivative is given by

$$u'_\alpha(x) = x^{\alpha-1},$$

which is positive for all x (since $0 < \alpha < 1$), so these functions are increasing. To show concavity, we examine the second derivative:

$$u''_\alpha(x) = (\alpha - 1)x^{\alpha-2}.$$

Since this is negative for all x, we conclude that u_α is strictly concave.

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

Solution. Strictly convex, and therefore quasiconvex. Also quasiconcave but not concave.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is neither positive semidefinite nor negative semidefinite. Therefore, f is neither convex nor concave. It is quasiconcave, since its superlevel sets

$$\{(x_1, x_2) \in \mathbf{R}^2_{++} \mid x_1 x_2 \geq \alpha\}$$

are convex. It is not quasiconvex.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

Solution. The Hessian of f is

$$\nabla^2 f(x) = \frac{1}{x_1 x_2} \begin{bmatrix} 2/(x_1^2) & 1/(x_1 x_2) \\ 1/(x_1 x_2) & 2/x_2^2 \end{bmatrix} \succeq 0$$

Therefore, f is convex and quasiconvex. It is not quasiconcave or concave.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

Solution. The Hessian of f is

$$\nabla^2 f(x) = \begin{bmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{bmatrix}$$

which is not positive or negative semidefinite. Therefore, f is not convex or concave.

It is quasiconvex and quasiconcave (*i.e.*, quasilinear), since the sublevel and superlevel sets are halfspaces.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

Solution. f is convex, as mentioned on page 72. (See also figure 3.3). This is easily verified by working out the Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} 2/x_2 & -2x_1/x_2^2 \\ -2x_1/x_2^2 & 2x_1^2/x_2^3 \end{bmatrix} = (2/x_2) \begin{bmatrix} 1 \\ -x_1/x_2 \end{bmatrix} \begin{bmatrix} 1 & -x_1/x_2 \end{bmatrix} \succeq 0.$$

Therefore, f is convex and quasiconvex. It is not concave or quasiconcave (see the figure).

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$, on $\mathbf{R}^2_{++}$.

Solution. Concave and quasiconcave. The Hessian is

$$\begin{aligned} \nabla^2 f(x) &= \begin{bmatrix} \alpha(\alpha-1)x_1^{\alpha-2}x_2^{1-\alpha} & \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} \\ \alpha(1-\alpha)x_1^{\alpha-1}x_2^{-\alpha} & (1-\alpha)(-\alpha)x_1^\alpha x_2^{-\alpha-1} \end{bmatrix} \\ &= \alpha(1-\alpha)x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} -1/x_1^2 & 1/x_1x_2 \\ 1/x_1x_2 & -1/x_2^2 \end{bmatrix} \\ &= -\alpha(1-\alpha)x_1^\alpha x_2^{1-\alpha} \begin{bmatrix} 1/x_1 \\ -1/x_2 \end{bmatrix} \begin{bmatrix} 1/x_1 \\ -1/x_2 \end{bmatrix}^T \\ &\preceq 0. \end{aligned}$$

f is not convex or quasiconvex.

3.32 *Products and ratios of convex functions.* In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbf{R}$. Prove the following.

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

Solution.

(a) We prove the result by verifying Jensen's inequality. f and g are positive and convex, hence for $0 \leq \theta \leq 1$,

$$\begin{aligned} f(\theta x + (1-\theta)y)\, g(\theta x + (1-\theta)y) &\leq (\theta f(x) + (1-\theta)f(y))\,(\theta g(x) + (1-\theta)g(y)) \\ &= \theta f(x)g(x) + (1-\theta)f(y)g(y) \\ &\quad + \theta(1-\theta)(f(y)-f(x))(g(x)-g(y)). \end{aligned}$$

The third term is less than or equal to zero if f and g are both increasing or both decreasing. Therefore

$$f(\theta x + (1-\theta)y)\, g(\theta x + (1-\theta)y) \leq \theta f(x)g(x) + (1-\theta)f(y)g(y).$$

(b) Reverse the inequalities in the solution of part (a).

(c) It suffices to note that $1/g$ is convex, positive and increasing, so the result follows from part (a).

3.36 Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.
Solution. We will show that

$$f^*(y) = \begin{cases} 0 & \text{if } y \succeq 0,\ \ \mathbf{1}^T y = 1 \\ \infty & \text{otherwise.} \end{cases}$$

We first verify the domain of f^*. First suppose y has a negative component, say $y_k < 0$. If we choose a vector x with $x_k = -t$, $x_i = 0$ for $i \neq k$, and let t go to infinity, we see that

$$x^T y - \max_i x_i = -t y_k \to \infty,$$

so y is not in $\mathbf{dom}\, f^*$. Next, assume $y \succeq 0$ but $\mathbf{1}^T y > 1$. We choose $x = t\mathbf{1}$ and let t go to infinity, to show that

$$x^T y - \max_i x_i = t\mathbf{1}^T y - t$$

is unbounded above. Similarly, when $y \succeq 0$ and $\mathbf{1}^T y < 1$, we choose $x = -t\mathbf{1}$ and let t go to infinity.

The remaining case for y is $y \succeq 0$ and $\mathbf{1}^T y = 1$. In this case we have

$$x^T y \leq \max_i x_i$$

for all x, and therefore $x^T y - \max_i x_i \leq 0$ for all x, with equality for $x = 0$. Therefore $f^*(y) = 0$.

(c) *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

Solution. Under the assumption, the graph of f is piecewise-linear, with breakpoints $(b_i - b_{i+1})/(a_{i+1} - a_i)$, $i = 1, \ldots, m-1$. We can write f^* as

$$f^*(y) = \sup_x \left(xy - \max_{i=1,\ldots,m}(a_i x + b_i) \right)$$

We see that $\mathbf{dom}\, f^* = [a_1, a_m]$, since for y outside that range, the expression inside the supremum is unbounded above. For $a_i \leq y \leq a_{i+1}$, the supremum in the definition of f^* is reached at the breakpoint between the segments i and $i+1$, *i.e.*, at the point $(b_{i+1} - b_i)/(a_{i+1} - a_i)$, so we obtain

$$f^*(y) = -b_i - (b_{i+1} - b_i)\frac{y - a_i}{a_{i+1} - a_i}$$

where i is defined by $a_i \leq y \leq a_{i+1}$. Hence the graph of f^* is also a piecewise-linear curve connecting the points $(a_i, -b_i)$ for $i = 1, \ldots, m$. Geometrically, the epigraph of f^* is the epigraphical hull of the points $(a_i, -b_i)$.

4.1 Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$.

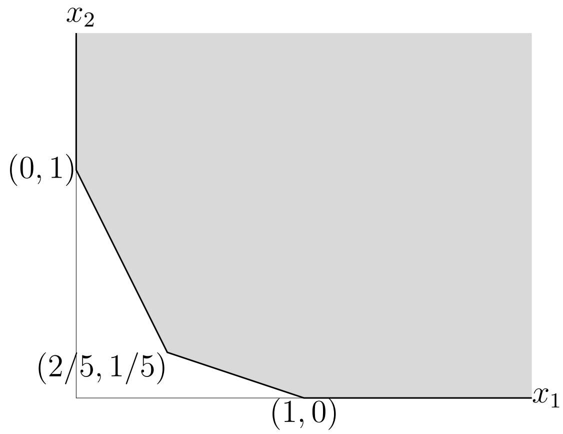
(b) $f_0(x_1, x_2) = -x_1 - x_2$.

(c) $f_0(x_1, x_2) = x_1$.

(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$.

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Solution. The feasible set is shown in the figure.



(a) $x^\star = (2/5, 1/5)$.

(b) Unbounded below.

(c) $X_{\rm opt} = \{(0, x_2) \mid x_2 \geq 1\}$.

(d) $x^\star = (1/3, 1/3)$.

(e) $x^\star = (1/2, 1/6)$. This is optimal because it satisfies $2x_1 + x_2 = 7/6 > 1$, $x_1 + 3x_2 = 1$, and

$$\nabla f_0(x^\star) = (1, 3)$$

is perpendicular to the line $x_1 + 3x_2 = 1$.

Solutions to additional exercises

1. *'Hello World' in CVX.* Use CVX to verify the optimal values you obtained (analytically) for exercise 4.1.

   Solution.

   (a) $p^\star = 0.6$

   (b) $p^\star = -\infty$

   (c) $p^\star = 0$

   (d) $p^\star = \frac{1}{3}$

   (e) $p^\star = \frac{1}{2}$

```
%EX 4.1 using CVX
clear all
close all

%Set up the constraint parameters such that Ax<=b
A=[2 1;1 3;1 0;0 1];
b=[1 1 0 0]';

%set up a vector to store all the optimal values
optimal_values=zeros(5,1);

%Solve the 5 parts in order, using cvx:

%part a
%enter cvx problem environment
cvx_begin
%declare our optimization variable and size
variable x(2,1)
%we wish to minimize our objective function
minimize(sum(x))
%enter the constraints
A*x>=b;
%call to cvx to solve the active problem
cvx_end
optimal_values(1)=cvx_optval;


%part b
cvx_begin
```

```
variable x(2,1)
minimize(-sum(x))
A*x>=b;
cvx_end
optimal_values(2)=cvx_optval;


%part c
cvx_begin
variable x(2,1)
minimize(x(1))
A*x>=b;
cvx_end
optimal_values(3)=cvx_optval;


%part d
cvx_begin
variable x(2,1)
minimize(max(x))
A*x>=b;
cvx_end
optimal_values(4)=cvx_optval;


%part e
%this is a quadratic function, define pos def matrix C
C=[1 0;0 9];
cvx_begin
variable x(2,1)
minimize(x'*C*x)
A*x>=b;
cvx_end
optimal_values(5)=cvx_optval;
```

2. *Continued fraction function.* Show that the function

$$f(x) = \cfrac{1}{x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cfrac{1}{x_4}}}}$$

defined where every denominator is positive, is convex and decreasing. (There is nothing special about $n = 4$ here; the same holds for any number of variables.)

Solution. We will use the composition rules and recursion. $g_4(x) = 1/x_4$ is clearly convex and decreasing in x_4. The function $\frac{1}{x_3 - z}$ is convex in x_3 and z (over $\mathbf{dom}\, f$), and is decreasing in x_3 and increasing in z; it follows by the composition rules that $g_3(x) = \frac{1}{x_3 - g_4(x)}$ is convex and decreasing in all its variables. Repeating this argument for $g_k(x) = \frac{1}{x_k - g_{k+1}(x)}$ shows that f is convex and decreasing.

3. *Dual of intersection of cones.* Let C and D be closed convex cones in $\mathbf{R}^n$. In this problem we will show that

$$(C \cap D)^* = C^* + D^*.$$

Here, $+$ denotes set addition: $C^* + D^*$ is the set $\{u + v \mid u \in C^*,\ v \in D^*\}$. In other words, the dual of the intersection of two closed convex cones is the sum of the dual cones.

(a) Show that $C \cap D$ and $C^* + D^*$ are convex cones. (In fact, $C \cap D$ and $C^* + D^*$ are closed, but we won't ask you to show this.)

(b) Show that $(C \cap D)^* \supseteq C^* + D^*$.

(c) Now let's show $(C \cap D)^* \subseteq C^* + D^*$. You can do this by first showing

$$(C \cap D)^* \subseteq C^* + D^* \iff C \cap D \supseteq (C^* + D^*)^*.$$

You can use the following result:

If K is a closed convex cone, then $K^{**} = K$.

Next, show that $C \cap D \supseteq (C^* + D^*)^*$ and conclude $(C \cap D)^* = C^* + D^*$.

(d) Show that the dual of the polyhedral cone $V = \{x \mid Ax \succeq 0\}$ can be expressed as

$$V^* = \{A^T v \mid v \succeq 0\}.$$

Solution.

(a) Suppose $x \in C \cap D$. This implies that $x \in C$ and $x \in D$, which implies $\theta x \in C$ and $\theta x \in D$ for any $\theta \geq 0$. Thus, $\theta x \in C \cap D$ for any $\theta \geq 0$, which shows $C \cap D$ is a cone. We know $C \cap D$ is convex since the intersection of convex sets is convex.

To show $C^* + D^*$ is a closed convex cone, note that both C^* and D^* are convex cones, thus $C^* + D^*$ is the conic hull of $C^* \cup D^*$, which is a convex cone.

(b) Suppose $x \in C^* + D^*$. We can write x as $x = u + v$, where $u \in C^*$ and $v \in D^*$. We know $u^T y \geq 0$ for all $y \in C$ and $v^T y \geq 0$ for all $y \in D$, which implies that $x^T y = u^T y + v^T y \geq 0$ for all $y \in C \cap D$. This shows $x \in (C \cap D)^*$, and so $(C \cap D)^* \supseteq C^* + D^*$.

(c) We showed in part (a) that $C \cap D$ and $C^* + D^*$ are closed convex cones. This implies $(C \cap D)^{**} = C \cap D$ and $(C^* + D^*)^{**} = (C^* + D^*)$, and so

$$(C \cap D)^* \subseteq C^* + D^* \iff C \cap D \supseteq (C^* + D^*)^*.$$

Suppose $x \in (C^* + D^*)^*$. $x^T y \geq 0$ for all $y = u + v$, where $u \in C^*$, $v \in D^*$. This can be written as $x^T u + x^T v \geq 0$, for all $u \in C^*$ and $v \in D^*$. Since $0 \in C^*$ and $0 \in D^*$, taking $v = 0$ we get $x^T u \geq 0$ for all $u \in C^*$, and taking $u = 0$ we get $x^T v \geq 0$ for all $v \in D^*$. This implies $x \in C^{**} = C$ and $x \in D^{**} = D$, *i.e.*, $x \in C \cap D$.

So we have shown both $(C \cap D)^* \supseteq C^* + D^*$ and $(C \cap D)^* \subseteq C^* + D^*$, which implies $(C \cap D)^* = C^* + D^*$.

(d) Using the result we just proved, we can write

$$V^* = \{x \mid a_1^T x \geq 0\}^* + \cdots + \{x \mid a_m^T x \geq 0\}^*.$$

The dual of $\{x \mid a_i^T x \geq 0\}$ is the set $\{\theta a_i \mid \theta \geq 0\}$, so we get

$$\begin{aligned} V^* &= \{\theta a_1 \mid \theta \geq 0\} + \cdots + \{\theta a_m \mid \theta \geq 0\} \\ &= \{\theta_1 a_1 + \cdots + \theta_m a_m \mid \theta_i \geq 0,\ i = 1, \ldots, m\}. \end{aligned}$$

This can be more compactly written as

$$V^* = \{A^T v \mid v \succeq 0\}.$$