Review Session 9
Perron-Frobenius

- Power method
- Google page ranking
- Irreducible matrices
- Heat conduction problem
- Circuit analysis

Power method

consider $x(t+1) = Ax(t)$ with $A \geq 0$ and regular then by PF theorem, λ_{pf} is the unique dominant eigenvalue

if v, $w > 0$ are left and right PF eigenvectors of A, with $\mathbf{1}^T v = 1$, $w^T v = 1$
then as $t \rightarrow \infty, (\lambda_{\text{pf}}^{-1} A)^t \rightarrow v w^T$
for any $x(0) \geq 0$, $x(0) \neq 0$, we have

$$\frac{1}{\mathbf{1}^T x(t)} x(t) \rightarrow v$$

and $x_i(t+1)/x_i(t) \rightarrow \lambda_{\text{pf}}$ as $t \rightarrow \infty$

this gives a way to compute λ_{pf} and v

this idea can be extended
suppose that $A \in \mathbf{R}^{n\times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$$

if v, w are left and right eigenvectors of λ_1, with $\mathbf{1}^T v = 1$, $w^T v = 1$

then as $t \to \infty, (\lambda_1^{-1} A)^t \to v w^T$
for any $x(0)$ such that $w^T x(0) \neq 0$, we have

$$\frac{1}{\mathbf{1}^T x(t)} x(t) \to v$$

and $x_i(t+1)/x_i(t) \to \lambda_1$ as $t \to \infty$

Rate of convergence

assuming A diagonalizable we have

$$(\lambda_1^{-1}A)^t = T(\lambda_1^{-1}\Lambda)^t T^{-1} = T \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & (\lambda_2/\lambda 1)^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (\lambda_n/\lambda_1)^t \end{bmatrix} T^{-1}$$

rate of convergence is λ_2/λ_1

Shifted power method

suppose $A \in \mathbf{R}^{n \times n}$ has distinct eigenvalues
we have

$$\max \lambda((A - \alpha I)^{-1}) = \max_i \frac{1}{\lambda_i - \alpha}$$

for any k by picking α we can make $\frac{1}{\lambda_k - \alpha}$ the dominant eigenvalue

- we can use the previous technique to compute $\frac{1}{\lambda_k - \alpha}$ and therefore λ_k
- we compute the λ closest to α

Google page ranking

suppose that a_i for $i = 1, \ldots, n$ represent all the web pages of the web

let's define $B \in \mathbf{R}^{n \times n}$ so that

$$B_{ij} = \begin{cases} 1 & \text{if } a_j \text{ contains a link to } a_i \\ 0 & \text{otherwise} \end{cases}$$

and P such that

$$P_{ij} = \frac{B_{ij}}{\sum_i B_{ij}}$$

P is a stochastic matrix

assuming P is regular

- $\lambda_{\text{pf}} = 1$ is the dominant eigenvalue
- we can find the π stable distribution $P\pi = \pi$
- π_j is used as the rank of pages a_j

Interpretations

for $p(t+1) = Pp(t)$

- we can interpret $p(t)$ has the scores given to the pages at time t
  at the next step page j gives a score to all the pages linked by it so that the sum of these scores is $p_j(t)$
  the next score is obtained by summing the score given by all the pages
  π is the limit score

- we can also interpret $p(t)$ has the probability of a user to visit the different pages at time t
  $p(t+1)$ is the probability of visiting the pages assuming that we randomly pick one page among the one linked by the current one
  π is the limit probability distribution

- if P is regular to compute π a power method can be used
  for $x(t) = P^t x(0)$
  $x(t) \to \pi$ as $t \to \infty$
  (this is not the method used in practice)

- if P is not regular we can use

$$\tilde{B} = B + \alpha \mathbf{1}\mathbf{1}^T,$$

- how fast it converges?
  the original paper on Google claims for 24 million pages 50 iterations are required

Irreducible matrices

given $A \in \mathbf{R}^{n\times n} \geq 0$ we say it's irreducible if for any i, j there is k such that

$$A^k_{ij} > 0$$

irreducible $\not\Rightarrow$ regular

regular $\Rightarrow$ irreducible

A irreducible implies $A+I$ is regular

in fact for each ij let's define k_{ij} so that

$$A_{ij}^{k_{ij}} > 0$$

and $k = \max_{ij} k_{ij}$ we have

$$(A+I)^k = A^k + \beta_{k-1}A^{k-1} + \cdots + \beta_1 A + I,$$

where $\beta_i > 0$ and therefore

$$(A+I)^k > 0$$

- we are introducing self loops for each node
- therefore if we can reach a node in k steps for $m > k$ we can reach it in m steps

Heat conduction problem

suppose we want to numerically solve the heat conduction problem

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial T}{\partial x^2}$$

for $T \in (a, b)$ and $T(x, 0) = f(x)$
α is the thermal diffusivity

we discretize x in n intervals with $x_i = a + i\Delta x$ with $\Delta x = (b - a)/n$
we discretize time with $t_i = i\Delta t$

we define $\tilde{T}(j) \in \mathbf{R}^n$ so that

$$\tilde{T}(j)_i = T(x_i, t_j)$$

a famous discretization of the partial differential equation is given by

$$\tilde{T}(j+1) = A\tilde{T}(j)$$

where

$$A = \begin{bmatrix} 1-2h & h & 0 & \cdots & 0 & 0 \\ h & 1-2h & h & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & \cdots & h & 1-2h & h \\ 0 & 0 & \cdots & 0 & h & 1-2h \end{bmatrix}$$

and $h = \alpha \Delta t / \Delta x^2$ and $T(0)_i = f(x_i)$

for $h < 1/2$, $A \geq 0$ therefore there there is a dominant λ_{pf} and $v \geq 0$ with

$$\lambda_{\mathrm{pf}} v = Av$$

we then have

$$\lambda_{\mathrm{pf}} v_i = h v_{i-1} + (1 - 2h) v_i + h v_{i+1}$$

where $v_{-1} = v_{n+1} = 0$

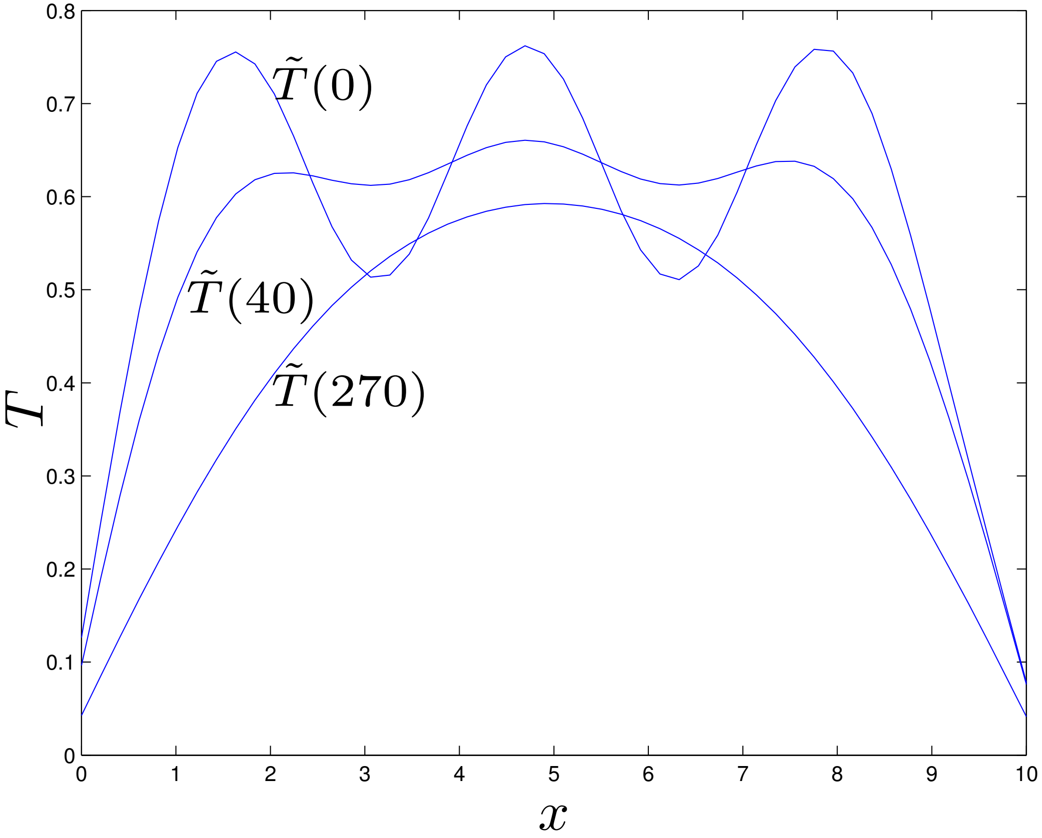
if k is the first index for which v_i is maximum we have

$$\lambda_{\mathrm{pf}} = h \frac{v_{i-1}}{v_i} + (1 - 2h) + h \frac{v_{i+1}}{v_i} < 1$$

and therefore the system is stable and the solution will converge to 0

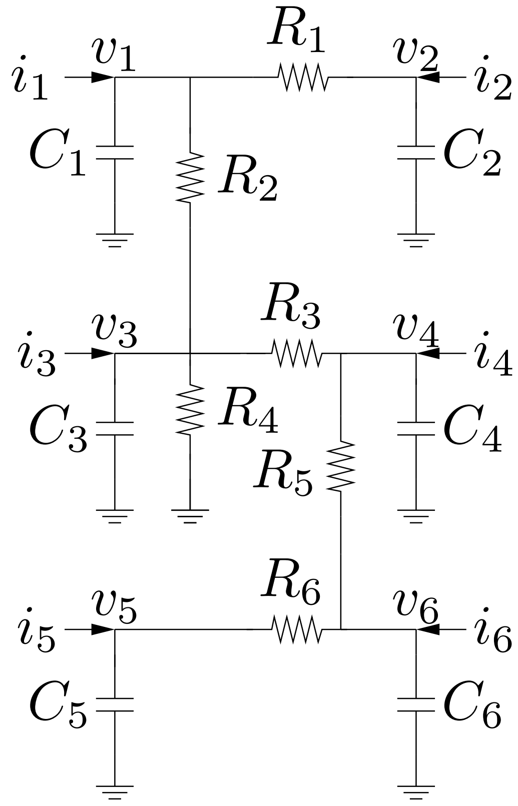
Numerical example

For $\alpha = 0.1$, $a = 0$, $b = 10$, $n = 50$, $\Delta t = 0.1$ we have



Circuit analysis

consider a circuit with n nodes formed only by capacitors and resistors with an injected current at each node, capacitors only between the nodes and ground
for example



we can write

$$C\dot{v} + Gv - i = 0,$$

- C is a diagonal matrix with $C_{ii} = C_i$
- $-G_{ij}$ is conductance between node i and node j for $i \neq j$
- G_{ii} is sum of the conductance connected to node i
- $i = (i_1, \ldots, i_n)$

for the specific example in the figure we have

$$C = \mathbf{diag}(C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6),$$

$$G = \begin{bmatrix} \frac{1}{R_1}+\frac{1}{R_2} & -\frac{1}{R_1} & -\frac{1}{R_2} & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1} & 0 & 0 & 0 & 0 \\ -\frac{1}{R_2} & 0 & +\frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_3} & 0 & 0 \\ 0 & 0 & -\frac{1}{R_3} & +\frac{1}{R_3}+\frac{1}{R_5} & 0 & -\frac{1}{R_5} \\ 0 & 0 & 0 & 0 & \frac{1}{R_6} & -\frac{1}{R_6} \\ 0 & 0 & 0 & -\frac{1}{R_5} & -\frac{1}{R_6} & \frac{1}{R_6} \end{bmatrix}$$

we have

$$\dot{V} = -C^{-1}GV - i$$

- since $-C^{-1/2}GC^{-1/2}$ is symmetric all the eigenvalues are real in fact using a symilarity transformation we have $-C^{1/2}C^{-1}GC^{-1/2} = -C^{-1/2}GC^{-1/2}$

- $-C^{-1}G$ and $-G$ are Metzler matrices

- there is $\lambda_{\mathrm{metzler}} \in \mathbf{R}$ with $v \geq 0$, $\lambda_{\mathrm{metzler}} v = -C^{-1}Gv$, and $\Re\lambda \leq \lambda_{\mathrm{metzler}}$

- if G is nonsingular $\lambda_{\mathrm{metzler}} < 0$, the system is stable

- v is the dominant mode, if we charge the capacitors with voltage v they all discharge at the same rate $\lambda_{\mathrm{metzler}}$

proof

since $\sum_{i\neq k}(-C^{-1}G)_{ki} \leq (C^{-1}G)_{kk}$ if we pick k such that $v_k \geq v_i$, $i = 1, \ldots, n$

$$\lambda_{\mathrm{metzler}} = -(C^{-1}G)_{kk} - \sum_{i\neq k}(-C^{-1}G)_{ki}\frac{v_i}{v_k} \leq 0$$

since C is nonsingular, $-C^{-1}G$ is nonsingular and therefore $\lambda_{\mathrm{metzler}} < 0$

DC analysis

we know the system is stable, we are interested in the stable point

$$v_0 = G^{-1}i$$

these are the voltages after reaching convergence

- $(\alpha I + G)^{-1} \geq 0$ for $\alpha > \lambda_{\mathrm{metzler}}$
- in particular $G^{-1} \geq 0$
- if $i > 0$ the stable point $v_0 \geq 0$

this makes sense, for positive injected currents we obtain only nonnegative voltages