EE 359 - Wireless Communications - Fall 2009

Homework 3 Solutions

1. (3-13) (5 pts)

   In the reader, we found the level crossing rate below a level by taking an average of the number of times the level was crossed over a large period of time. It is easy to convince that the level crossing rate above a level will have the same expression as eq. 3.44 in reader because to go below a level again, we first need to go above it. There will be some discrepancy at the end points, but for a large enough T it will not effect the result. So we have

   $$L_Z(above) = L_Z(below) = \sqrt{2\pi} f_D \rho e^{-\rho^2}$$

   And,

   $$\overline{t_Z}(above) = \frac{p(z > Z)}{L_Z(above)}$$

   $$p(z > Z) = 1 - p(z \leq Z) = 1 - (1 - e^{-\rho^2}) = e^{-\rho^2}$$

   $$\overline{t_Z}(above) = \frac{1}{\sqrt{2\pi} f_D \rho}$$

   (3 pts)

   The values of $\overline{t_Z}(above)$ for f_D = 10,50,80 Hz are 0.0224s, 0.0045s, 0.0028s respectively. Notice that as f_D increases, $\overline{t_Z}(above)$ reduces. (1pt)

2. (3-14) (10 pts)
   For $T = 10\mu s$
   $\bar{P}_r = 10dBm.$
   $f_D = 80Hz$
   $N = 1dBm.$

   In the following computations, make sure that A_j, A_{j+1} and $\bar{P}_r$ are converted in the same unit, Watt or mWatt. Using results on Page 82 in the text,

   $$p_{j,j+1} = \frac{L_{j+1}T}{\pi_j}, \quad p_{j,j-1} = \frac{L_j T}{\pi_j}, \quad p_{j,j} = 1 - p_{j,j+1} - p_{j,j-1}.$$

   where

   $$L_j = \sqrt{2\pi} f_D \rho e^{-\rho^2}, \quad \rho = \frac{A_j N}{\sqrt{\bar{P}_r}} \quad \text{level crossing rate at } A_j.$$

   So we need two set of quantities: L_js and π_js.

   Note that for Rayleigh fading channel, γ is Rayleigh distributed:

   $$\begin{aligned} \pi_j &= P(A_j \leq \gamma < A_{j+1}) && (1)\\ &= P(A_j N \leq P \leq A_{j+1}N) \quad (\gamma N = P, \quad P \text{ is the signal power}) && (2)\\ &= \int_{A_j N}^{A_{j+1}N} \frac{1}{\bar{P}_r} e^{-\frac{x}{\bar{P}r}} dx && (3)\\ &= e^{-A_j N/\bar{P}_r} - e^{-A_{j+1}N/\bar{P}_r}. && (4) \end{aligned}$$

$$
\begin{aligned}
R_1:&\quad -\infty \le \gamma \le -10dB, && \pi_1 = 9.95\times10^{-3}\\
R_2:&\quad -10dB \le \gamma \le 0dB, && \pi_2 = 0.085\\
R_3:&\quad 0dB \le \gamma \le 5dB, && \pi_3 = 0.176\\
R_4:&\quad 5dB \le \gamma \le 10dB, && \pi_4 = 0.361\\
R_5:&\quad 10dB \le \gamma \le 15dB, && \pi_5 = 0.325\\
R_6:&\quad 15dB \le \gamma \le 20dB, && \pi_6 = 0.042\\
R_7:&\quad 20dB \le \gamma \le 30dB, && \pi_7 = 4.54\times10^{-5}\\
R_8:&\quad 30dB \le \gamma \le \infty, && \pi_8 = 3.72\times10^{-44}
\end{aligned}
$$

(get correct π_js, 4 pts)

$L_j \rightarrow$ level crossing rate at level A_j

$$
\begin{aligned}
L_1 &= 0, & \rho &= \sqrt{\tfrac{0}{10}}\\
L_2 &= 19.85, & \rho &= \sqrt{\tfrac{0.1}{10}}\\
L_3 &= 57.38, & \rho &= \sqrt{\tfrac{1}{10}}\\
L_4 &= 82.19, & \rho &= \sqrt{\tfrac{10^{0.5}}{10}}\\
L_5 &= 73.77, & \rho &= \sqrt{\tfrac{10}{10}}\\
L_6 &= 15.09, & \rho &= \sqrt{\tfrac{10^{1.5}}{10}}\\
L_7 &= 0.03, & \rho &= \sqrt{\tfrac{10^{2}}{10}}\\
L_8 &= 0, & \rho &= \sqrt{\tfrac{10^{3}}{10}}
\end{aligned}
$$

(get correct L_js, 4 pts)

(get correct transition probabilities, using the right formula, 2 pts)

```
MATLAB CODE: L = [0 19.85 57.38 82.19 73.77 15.09 .03 0];

Pi = [9.95e-3 .085 .176 .361 .325 .042 4.54e-5 3.72e-44];

T = 10e-6; for i = 1:8
    if i == 1
        p(i,1) = 0;
        p(i,2) = (L(i+1)*T)/Pi(i); % i to i+1
        p(i,3) = 1-(p(i,1)+p(i,2)); % i to i-1
    elseif i == 8
        p(i,1) = (L(i)*T)/Pi(i);
        p(i,2) = 0;
        p(i,3) = 1-(p(i,1)+p(i,2));
    else
        p(i,1) = (L(i)*T)/Pi(i);
        p(i,2) = (L(i+1)*T)/Pi(i);
        p(i,3) = 1-(p(i,1)+p(i,2));
    end
end
```

```
% p =
%
%            0    0.0199    0.9801
%       0.0023    0.0068    0.9909
%       0.0033    0.0047    0.9921
%       0.0023    0.0020    0.9957
%       0.0023    0.0005    0.9973
%       0.0036    0.0000    0.9964
%       0.0066         0    0.9934
%            0         0    1.0000
```

3. (3-15) (12 pts)

   (a) (5pts)

$$S(\tau,\rho) = \begin{cases} \alpha_1\delta(\tau) & \rho = 70Hz \\ \alpha_2\delta(\tau - 0.022\mu sec) & \rho = 49.5Hz \\ 0 & else \end{cases}$$

   The antenna setup is shown in Fig. 3
   From the figure, the distance travelled by the LOS ray is d and the distance travelled by the first multipath component is

$$2\sqrt{\left(\frac{d}{2}\right)^2 + 64}$$

   Given this setup, we can plot the arrival of the LOS ray and the multipath ray that bounces off the ground on a time axis as shown in Fig. 3
   So we have

$$2\sqrt{\left(\frac{d}{2}\right)^2 + 8^2} - d = 0.022 \times 10^{-6} 3 \times 10^8$$

$$\Rightarrow 4\left(\frac{d^2}{4} + 8^2\right) = 6.6^2 + d^2 + 2d(6.6)$$

$$\Rightarrow d = 16.1m$$

   $f_D = v\cos(\theta)/\lambda$. $v = f_D\lambda/\cos(\theta)$. For the LOS ray, $\theta = 0$ and for the multipath component, $\theta = 45^o$. We can use either of these rays and the corresponding f_D value to get $v = 23.33$m/s.

   (b) (4pts)

$$d_c = \frac{4h_t h_r}{\lambda}$$

   $d_c = 768$ m. Since $d \ll d_c$, power fall-off is proportional to d^{-2}.

   (c) (3pts)$T_m = 0.022\mu$s, $B^{-1} = 0.33\mu$s. Since $T_m \ll B^{-1}$, we have flat fading.

4. (3-16) (12 pts)

   (a) (1 pt) Outdoor, since delay spread ≈ 10 μsec.
   Consider that 10 μsec $\Rightarrow$ d = ct = 3km difference between length of first and last path

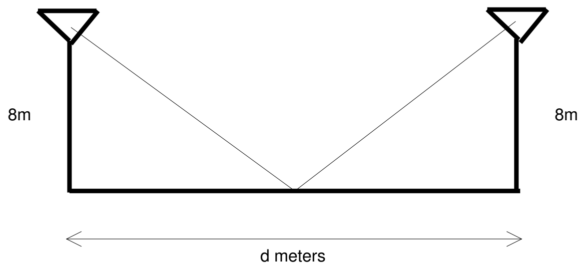

Figure 1: Problem 3-15: Antenna Setting

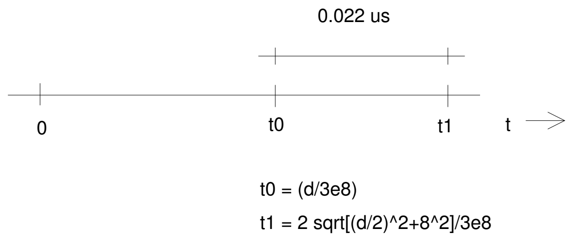

Figure 2: Problem 3-15: Time Axis for Ray Arrival

(b) (1 pt)
Scattering function

$$S(\tau,\rho) = F_{\Delta t}[A_c(\tau,\Delta t)]$$

$= \frac{1}{W} rect\left(\frac{1}{W}\rho\right)$ for $0 \le \tau \le 10\mu sec$

The Scattering function is plotted in Fig. 4

(c) (2 pts) Avg Delay Spread $= \frac{\int_0^\infty \tau A_c(\tau)d\tau}{\int_0^\infty A_c(\tau)d\tau} = 5\mu\text{sec}$

RMS Delay Spread $= \sqrt{\frac{\int_0^\infty (\tau-\mu_{Tm})^2 A_c(\tau)d\tau}{\int_0^\infty A_c(\tau)d\tau}} = 2.89\mu\text{sec}$

Doppler Spread $= \frac{W}{2} = 50$ Hz

(d) (2 pts) $\beta_u >$ Coherence BW $\Rightarrow$ Freq. Selective Fading $\approx \frac{1}{T_m} = 10^5 \Rightarrow \beta_u >= 10^5$ kHz
Can also use μ_{Tm} or σ_{Tm} instead of T_m

(e) (2 pts) Rayleigh fading, since receiver power is evenly distributed relative to delay; no dominant LOS path

(f) (2 pts)$t_R = \frac{e^{\rho^2}-1}{\rho F_D\sqrt{2\pi}}$ with $\rho = 1$, $f_D = \frac{W}{2} \rightarrow\ t_r = .0137$ sec

(g) (2 pts) Notice that the fade duration never becomes more than twice the average. So, if we choose our data rate such that a single symbol spans the average fade duration, in the worst case two symbols will span the fade duration. So our code can correct for the lost symbols and we will have error-free transmission. So $\frac{1}{t_R} = 72.94$ symbols/sec

5. (4-2) (4 pts)
$B = 50$ MHz
$P = 10$ mW
$N_0 = 2\ \times 10^{-9}$ W/Hz

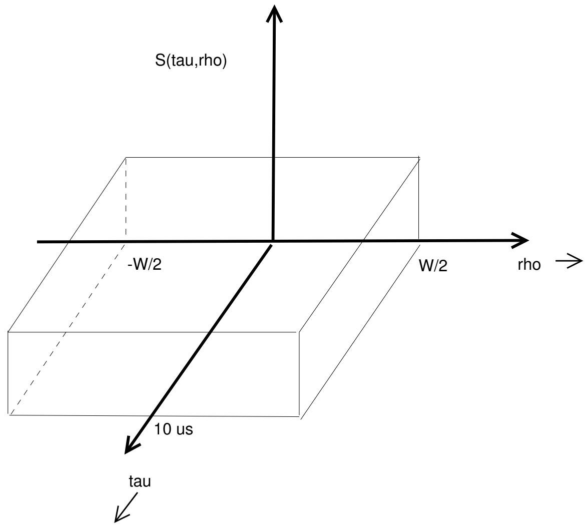

Figure 3: Problem 3-16: Scattering Function

$N = N_0 B$
$C = 6.8752$ Mbps.

$P_{\text{new}} = 20$ mW, $C = 13.1517$ Mbps, increase by 6.2765 Mbps. (2pts)
$B = 100$ MHz, Notice that both the bandwidth and noise power will increase. So $C = 7.0389$ Mbps, increase by 0.1637 Mbps. (2pts)

6. (4-5) (12 pts)

   (a) (4pts) We suppose that all channel states are used

$$\frac{1}{\gamma_0} = 1 + \sum_{i=1}^{4} \frac{1}{\gamma_i} p_i \Rightarrow \ \gamma_0 = 0.8109$$

$$\frac{1}{\gamma_0} - \frac{1}{\gamma_4} > 0 \therefore true$$

$$\frac{S(\gamma_i)}{\overline{S}} = \frac{1}{\gamma_0} - \frac{1}{\gamma_i}$$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 1.2322 & \gamma = \gamma_1 \\ 1.2232 & \gamma = \gamma_2 \\ 1.1332 & \gamma = \gamma_3 \\ 0.2332 & \gamma = \gamma_4 \end{cases}$$

$$\frac{C}{B} = \sum_{i=1}^{4} \log_2 \left(\frac{\gamma_i}{\gamma_0} \right) p(\gamma_i) = 5.2853 bps/Hz$$

   (b) (4pts) $\sigma = \frac{1}{E[1/\gamma]} = 4.2882$
   $\frac{S(\gamma_i)}{\overline{S}} = \frac{\sigma}{\gamma_i}$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 0.0043 & \gamma = \gamma_1 \\ 0.0029 & \gamma = \gamma_2 \\ 0.4288 & \gamma = \gamma_3 \\ 4.2882 & \gamma = \gamma_4 \end{cases}$$

$$\frac{C}{B} = \log_2(1+\sigma) = 2.4028 bps/Hz$$

(c) (4pts) To have $p_{out} = 0.1$ we will have to use all the sub-channels as leaving any of these will result in a p_{out} of at least 0.2 $\therefore$ truncated channel power control policy and associated spectral efficiency are the same as the zero-outage case in part b .

To have $p_{out} = 0.25$, remove the worst subchannel with $\gamma_4 = 0$dB, we have $P_{out} = 0.2 < 0.25$, $C/B = 3.9678 bps/Hz$.

To have p_{out} that maximizes C with truncated channel inversion, we try different truncation strategy: 1) remove the subchannel with $\gamma_4 = 0$dB, we have $P_{out} = 0.2$, $C/B = 3.9678 bps/Hz$; 2) remove 2 worse subchannels with $\gamma_4 = 0$dB, and $\gamma_3 = 10$dB, we have $C/B = 4.1462$ bps/Hz, $P_{out} = 0.5$; 3) remove three worse subchannels, $P_{out} = 0.8$, $C/B = 2.4576$ bps/Hz. Compare these cases and also that using all the subchannels, we get

$$\max \frac{C}{B} = 4.1462 bps/Hz \quad p_{out} = 0.5.$$

7. (4-7) (15pts)

(a) (3pts) Maximize capacity given by

$$C = \max_{S(\gamma): \int S(\gamma)p(\gamma)d\gamma = \overline{S}} \int_\gamma B \log\left(1 + \frac{S(\gamma)\gamma}{\overline{S}}\right) p(\gamma) d\gamma.$$

Construct the Lagrangian function

$$\mathcal{L} = \int_\gamma B \log\left(1 + \frac{S(\gamma)\gamma}{\overline{S}}\right) p(\gamma) d\gamma - \lambda \int \frac{S(\gamma)}{\overline{S}} p(\gamma) d\gamma$$

Taking derivative with respect to $S(\gamma)$, (refer to discussion section notes) and setting it to zero, we obtain,

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma} & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

Now, the threshold value must satisfy

$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p(\gamma) d\gamma = 1$$

Evaluating this with $p(\gamma) = \frac{1}{10} e^{-\gamma/10}$, we have

$$1 = \frac{1}{10\gamma_0} \int_{\gamma_0}^{\infty} e^{-\gamma/10} d\gamma - \frac{1}{10} \int_{\gamma_0}^{\infty} \frac{e^{-\gamma/10}}{\gamma} d\gamma \quad (5)$$

$$= \frac{1}{\gamma_0} e^{-\gamma_0/10} - \frac{1}{10} \int_{\frac{\gamma_0}{10}}^{\infty} \frac{e^{-\gamma}}{\gamma} d\gamma \quad (6)$$

$$= \frac{1}{\gamma_0} e^{-\gamma_0/10} - \frac{1}{10} \text{EXPINT}(\gamma_0/10) \quad (7)$$

where EXPINT is as defined in matlab. This gives $\gamma_0 = 0.7676$. The power adaptation becomes

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{0.7676} - \frac{1}{\gamma} & \gamma \geq 0.7676 \\ 0 & \gamma < 0.7676 \end{cases}$$

(b) (2pts) Capacity can be computed as

$$C/B = \frac{1}{10}\int_{0.7676}^{\infty} \log\left(\gamma/0.7676\right) e^{-\gamma/10} d\gamma = 2.0649 \ \ \text{nats/sec/Hz.} = 2.9790 \ \ \text{bits/sec/Hz.}$$

(In order to convert the capacity values from nats/sec/Hz to bits/sec/Hz, the capacity numbers simply need to be divided by natural log of 2.)

(c) (1 pts) AWGN capacity $C/B = \log(1 + 10) = 2.3979$ nats/sec/Hz $= 3.4594$ bits/sec/Hz.

(d) (2 pts) Capacity when only receiver knows γ

$$C/B = \frac{1}{10}\int_{0}^{\infty} \log\left(1+\gamma\right) e^{-\gamma/10} d\gamma = 2.0150 \ \ \text{nats/sec/Hz} = 2.9070 \ \ \text{bits/sec/Hz}$$

(e) (3 pts) Capacity using channel inversion is ZERO because the channel can not be inverted with finite average power. Threshold for outage probability 0.05 is computed as

$$\frac{1}{10}\int_{\gamma_0}^{\infty} e^{-\gamma/10} d\gamma = 0.95$$

which gives $\gamma_0 = 0.5129$. This gives us the capacity with truncated channel inversion as

$$\begin{aligned} C/B &= \log\left[1 + \frac{1}{\frac{1}{10}\int_{\gamma_0}^{\infty} \frac{1}{\gamma} e^{-\gamma/10} d\gamma}\right] * 0.95 && (8) \\ &= \log\left[1 + \frac{1}{\frac{1}{10}\text{EXPINT}(\gamma_0/10)}\right] * 0.95 && (9) \\ &= 1.5463 \quad \text{nats/s/Hz} = 2.2308 \quad \text{bits/s/Hz.} && (10) \end{aligned}$$

(f) (4 pts) Channel Mean=-5 dB = 0.3162. So for perfect channel knowledge at transmitter and receiver we compute $\gamma_0 = 0.22765$ which gives capacity $C/B = 0.36$ nats/sec/Hz $= 0.5194$ bits/sec/Hz.
With AWGN, $C/B = \log(1 + 0.3162) = 0.2748$ nats/sec/Hz $= 0.3963$ bits/sec/Hz.
With channel known only to the receiver $C/B = 0.2510$ nats/sec/Hz $= 0.3621$ bits/sec/Hz.

Capacity with AWGN is always greater than or equal to the capacity when only the reciever knows the channel. This can be shown using Jensen's inequality. However the capacity when the transmitter knows the channel as well and can adapt its power, can be higher than AWGN capacity specially at low SNR. At low SNR, the knowledge of fading helps to use the low SNR more efficiently.