EE359, Wireless Communications, Winter 2020
Homework 7 (100 pts)

Due: Friday (March 6), 4 pm

Please refer to the homework page on the website (`ee359.stanford.edu/homework`) for guidelines.

1. (15 pts) **SNR with Diversity** (Problem 10-13*): Consider a MIMO system where the channel gain matrix $\mathbf{H}$ is known at the transmitter and receiver.

   (a) (5 pts) Consider a 4x4 MIMO system operating in a channel with free space signal propagation between each transmit and receive antenna (i.e. there are no reflectors or scatterers in the environment). The antenna elements are collinear and the distance between antenna elements l at the receiver (or transmitter) is much smaller than the distance d between the antenna and the receiver as shown in Fig. 1. What is the approximate rank of the channel matrix $\mathbf{H}$ where singular values that are smaller than a threshold can be approximated to zero.

   (b) (10 pts) Show that if transmit and receive antennas are used for diversity, the optimal weights at the transmitter and receiver lead to an SNR of $\gamma = \lambda_{\max}\rho$, where $\lambda_{\max}$ is the largest eigenvalue of $\mathbf{HH}^H$. Also show that the leading eigenvectors of $\mathbf{H}^H\mathbf{H}$ and $\mathbf{HH}^H$ form the optimal transmit precoding and receiver shaping vectors respectively.

2. (15 pts) **Diversity order of maximum singular value**: In this question we look into the diversity order of the maximum singular value $\sigma_{\max}$. Consider an $m \times n$ matrix H with i.i.d. entries distributed as Rayleigh fading variables with power p each.

   (a) (4 pts) Use the definition of the Frobenius norm to establish an upper bound on the square of the maximum singular value of H in terms of the elements of H.

   (b) (4 pts) Use the fact that the maximum singular value is greater than all the singular values to come up with a lower bound for $\sigma_{\max}$.

   Hint: This is just a scaled version of the upper bound.

   (c) (7 pts) What is the diversity order of the Frobenius norm? Use this result and parts (a) and (b) to write down the diversity order for the maximum singular value. For any norm $\|\cdot\|$, we define

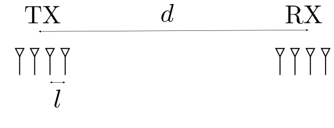

Figure 1

its diversity order to be

$$\lim_{p\to\infty} -\frac{\log(\text{Prob}(\|H\| < 1))}{\log(p)} .$$

Hint: Use results from the maximal ratio combining of Rayleigh fading random variables.

3. (10 pts) **On diversity multiplexing tradeoff** (Problem 10-15): Consider an 8×4 MIMO system, and assume a coding scheme that can achieve the rate-diversity tradeoff $d(r) = (M_t - r)(M_r - r)$.

   (a) Find the maximum multiplexing rate for this channel, given a required $P_e = \rho^{-d} \le 10^{-3}$ and assuming that $\rho = 10$ dB.

   (b) Given the r derived in part (a), what is the resulting P_e?

4. (20 pts) **MIMO systems and capacity** (Final 2012) An 8x8 MIMO system has a channel matrix $\mathbf{H}$ with singular values $\sigma_1 = 2, \sigma_2 = 1, \sigma_3 = 0.8, \sigma_4 = 0.3, \sigma_i = 0$ where $i = 5, 6, 7, 8$. Assume the total transmit power is $\rho = 10$ dBm, the noise power at each receive antenna is 0 dBm, and the system bandwidth is B = 10 MHz. Assume in this high-tech system that both transmitter and receiver know the channel matrix.

   (a) (4 pts) What is the rank of $\mathbf{H}$ and the SNR γ_i associated with each of its spatial dimensions assuming the full transmit power is allocated to it?

   (b) (4 pts) Find the Shannon capacity of the system assuming optimal adaptation of power and rate across spatial dimensions.

   (c) (4 pts) If the transmitter does not do power adaptation, it assigns equal power to all transmit antennas resulting in $\mathbf{R}_x = \frac{\rho}{8}\mathbf{I}$. Show that this implies that power is equally allocated over all 8 spatial dimensions. What is the capacity of the system in this case assuming that the rate is optimally adapted?

   (d) (4 pts) What is the capacity when only the signal dimension with the largest SNR is used, i.e. all power is allocated to the spatial dimension with the largest singular value (beamforming).

   (e) (4 pts) Find the maximum data rate that can be sent using adaptive modulation assuming power is optimally allocated across spatial dimensions. Compare with the maximum data rate in beamforming, where full power is allocated to the single spatial dimension associated with beamforming. In both cases assume a target BER of $P_b = 10^{-3}$ with unrestricted modulation (i.e. M can take any value). Assume a symbol time $T = 1/B$.

5. (15 pts) **MIMO receivers**: We are given the channel matrix $\mathbf{H}$, and a received vector $\mathbf{y}$, seen below.

$$\mathbf{H} = \begin{bmatrix} 0.3 & 0.5 & 0.5 \\ 0.7 & 0.4 & 0.4 \\ 0.4 & 0.5 & 0.6 \\ 0.2 & 0.6 & 0.7 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0.3 + 0.2i \\ 0.3 - 0.5i \\ -0.7 + 0..4i \\ -0.3 - 0.1i \end{bmatrix}$$

   Assume that the transmitter used all the antennas for multiplexing gain, and transmitted 3 QPSK symbols, where the constellation points for {(00),(01),(11),(10)} correspond to {(1+i),(-1+i),(-1-i),(1-i)} respectively. Estimate the transmitted sequence assuming the following MIMO receivers:

   (a) (5 pts) Maximum likelihood receiver

   (b) (5 pts) Zero-forcing receiver

   (c) (5 pts) Linear MMSE receiver (assume $N_0/E = 0.1$)

   Please attach any programming scripts you use.

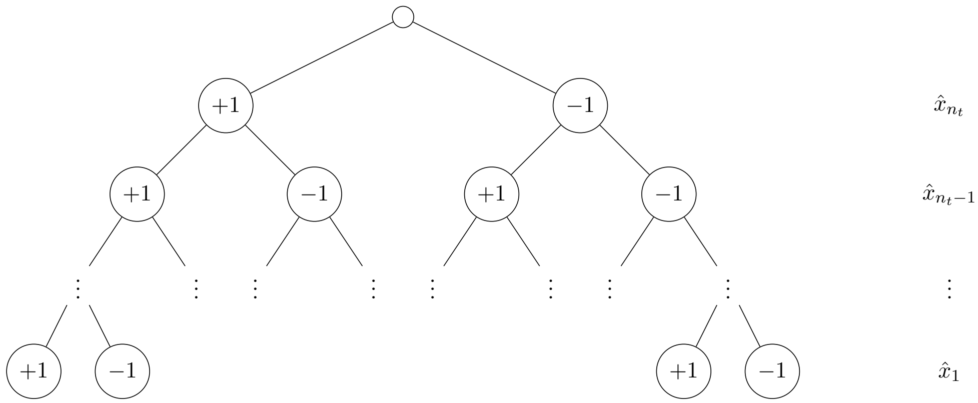

Figure 2: Search Tree for Problem 6

6. (15 pts) **Sphere Decoder.** In this problem, we will take a closer look at the sphere decoder. Consider that $\mathbf{H}$ is an $n_r \times n_t$ channel gain matrix and the $n_r \geq n_t$. For this problem we will assume that $\mathbf{H}$ is full rank. The channel gain matrix can be decomposed using the QR-decomposition as

$$\mathbf{H} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix},$$

where the columns of $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are all orthonormal and $\mathbf{R}$ is upper-triangular. Denote $\mathbf{y}$ as the n_r dimensional received vector and $\mathbf{x}$ as the n_t dimensional transmitted vector which consists of BPSK symbols.

   (a) (4 pts) State the dimensions of $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{R}$, and $\mathbf{0}$.

   (b) (3 pts) Suppose that at the receiver we are considering $\hat{\mathbf{x}}$ the transmitted signal. Show that if $\mathbf{H}\hat{\mathbf{x}}$ is within distance d of $\mathbf{y}$, then the following relation will hold

$$d^2 - \left\|\mathbf{Q}_2^H \mathbf{y}\right\|_2^2 \geq \left\|\mathbf{Q}_1^H \mathbf{y} - \mathbf{R}\hat{\mathbf{x}}\right\|_2^2.$$

   (c) (3 pts) Denote $\tilde{\mathbf{y}} = \mathbf{Q}_1^H \mathbf{y}$. Show that the previous condition can be written as

$$d^2 - \left\|\mathbf{Q}_2^H \mathbf{y}\right\|_2 \geq \sum_{i=n_t}^{1} \left(\tilde{y}_i - \sum_{j=i}^{n_t} R_{i,j} \hat{x}_j \right)^2. \qquad (*)$$

   Write the elements of the summation on the right-hand side of this expression corresponding to $i = n_t$ and $i = n_{t-1}$. In general, how many elements of $\hat{\mathbf{x}}$ are contained in the ith term of this summation?

   (d) (5 pts) We have previously seen that, for BPSK, the ML decoder must consider all 2^{n_t} possible transmitted vectors. The relation of $(*)$ gives us an intelligent way to search this space for the most-likely transmitted vector. Consider the tree given in Figure 2. Each root node on the bottom of this tree corresponds to a possible $\hat{\mathbf{x}}$, where each coordinate $\hat{x}_i$ is given by the parent node contained at layer i of the tree. That is, all root nodes that are childen of the node labeled '+1' at top layer labeled $\hat{x}_{n_t}$ have $\hat{x}_{n_t} = +1$. Describe how we can use the results of the last section to intelligently search this tree in order to find values of $\mathbf{H}\hat{\mathbf{x}}$ that are close to $\mathbf{y}$ without evalulating $\|\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}\|_2$ for all possible $\hat{\mathbf{x}}$.

7. (10 pts) **Minimum subcarrier separation** (Problem 12-1): Show that the minimum separation Δf for subcarriers $\{\cos(2\pi j \Delta f t + \phi_j\ ),\ j = 1, 2, \ldots\ \}$ to form a set of orthonormal basis functions on the interval $[0, T_N]$ is $\Delta f = 1/T_N$ for any initial phase ϕ_j . Show that if $\phi_j = 0$ for all j then this carrier separation can be reduced by half.