EE359, Wireless Communications, Winter 2020
Homework 6 (100 pts)

Due: Friday (February 28), 4 pm

Please refer to the homework page on the website (`ee359.stanford.edu/homework`) for guidelines.

1. (20 pts) **Optimal variable rate variable power MQAM** (Problem 9-9): Consider a discrete time-varying AWGN channel with four channel states. Assuming a fixed transmit power S, the received SNR associated with each channel state is $\gamma_1 = 5$ dB, $\gamma_2 = 10$ dB, $\gamma_3 = 15$ dB, and $\gamma_4 = 20$ dB, respectively. The probabilities associated with the channel states are $p(\gamma_1) = .4$ and $p(\gamma_2) = p(\gamma_3) = p(\gamma_4) = .2$. Also, the target $P_b = 10^{-3}$.
   (a) (10 pts) Find the optimal power and rate adaptation for continuous-rate adaptive MQAM on this channel.
   (b) (5 pts) Find the average spectral efficiency with this optimal adaptation.
   (c) (5 pts) Find the truncated channel inversion power control policy for this channel and the maximum data rate that can be supported with this policy.

2. (25 pts) **Adaptive MQAM with discrete constellations** (Problem 9-10, 9-11): Consider a Rayleigh fading channel with an average received SNR of 20 dB, a Doppler frequency of 80 Hz, and a required BER of 10^{-3} .
   (a) (10 pts) Find the spectral efficiency of this channel using truncated channel inversion, assuming the constellation is restricted to size 0, 2, 4, 16, 64, or 256. Please attach any programming script you write to solve this.
   (b) (10 pts) Suppose you use adaptive MQAM modulation on this channel with constellations restricted to size 0, 2, 4, 16, and 64. Using $\gamma_K^* = .1$, find the fading regions R_j associated with each of these constellations. Also find the average spectral efficiency of this restricted adaptive modulation scheme.
   (c) (5 pts) Does the data rate increase as γ_K^* increases? Does the transmit power associated with a given γ to meet the BER target increase or decrease as γ_K^* increases?

3. (15 pts) **Estimation error in adaptive modulation schemes** (Problem 9-12*): Consider a Rayleigh fading channel with an average received SNR of 20 dB, a signal bandwidth of 30 kHz, a Doppler frequency of 80 Hz, and a required BER of 10^{-3}. For this problem, please review Section 9.3.7 (Channel Estimation Error and Delay) of the text.
   (a) (5 pts) Assume the SNR estimate at the transmitter $\hat{\gamma}$ has the same distribution as the true channel SNR γ, so $p(\hat{\gamma}) = p(\gamma)$. For the optimal variable-rate variable-power MQAM scheme with no restrictions on rate or power, suppose that the transmit power and rate is adapted relative to $\hat{\gamma}$ instead of γ. Will the average transmitted data rate and average transmit power be larger, smaller, or the same as under perfect channel estimates ($\hat{\gamma} = \gamma$), and why? If over a given symbol time $\hat{\gamma} > \gamma$, will the probability of error associated with that symbol be larger or smaller than the target value and why?

(b) (5 pts) Denote the estimation error as $\epsilon = \hat{\gamma}/\gamma$, which is distributed according to some PDF $p(\epsilon)$. Give an expression for the bit error rate of the system in terms of the target BER, P_b^t and $p(\epsilon)$. You may find equation (9.34) from the reader helpful.

(c) (5 pts) Suppose the estimation error ϵ is uniformly distributed between .5 and 1.5. For a target BER of $P_b^t = 10^{-3}$, find the resulting average probability of bit error for this system.

4. (20 pts) **Equivalent MIMO capacity expressions** (Problems 10-5 & 10-7):

(a) (10 pts) The capacity of a static MIMO channel with only receiver CSI is given by $C = \sum_{i=1}^{R_{\mathbf{H}}} B \log_2(1+\sigma_i^2\rho/M_t)$. Show that, if the sum of squares of singular values (σ_i) is bounded, then this expression is maximized when all $R_{\mathbf{H}}$ singular values are equal.

(b) (10 pts) Use properties of the SVD to show that, for a MIMO channel that is known to the transmitter and receiver both, the general capacity expression

$$C = \max_{\mathbf{R_x}:Tr(\mathbf{R_x})=\rho} B \log_2 \det[\mathbf{I}_{M_r} + \mathbf{H}\mathbf{R_x}\mathbf{H}^{\mathbf{H}}]$$

reduces to

$$C = \max_{\rho_i:\sum_i \rho_i \leq \rho} \sum_i B \log_2(1 + \sigma_i^2 \rho_i)$$

for singular values $\{\sigma_i\}$ and SNR ρ.

Hint: The matrix $\mathbf{R_x}$ is the covariance matrix of the channel input distribution. It is positive semi-definite. You may find the following relation useful. For a positive definite matrix $\mathbf{A}$, then $\det(\mathbf{A}) \leq \prod_i A_{ii}$ where A_{ii} are the diagonal elements of $\mathbf{A}$. This equality holds if $\mathbf{A}$ is upper-triangular.

5. (20 pts) **Capacity of Massive-MIMO systems** (Problem 10-9 modified): Assume a ZMCSCG MIMO system with channel matrix $\mathbf{H} \in \mathbf{R}^{M_r \times M_t}$ corresponding to M_t transmit and M_r receive antennas ($\mathrm{E}[|H_{i,j}|^2] = 1$ for all i and j). Show using the law of large numbers that

$$\lim_{M_r \to \infty} \frac{1}{M_r} \mathbf{H}^H \mathbf{H} = \mathbf{I}_{M_t}.$$

Then use this to show that

$$\lim_{M_r \to \infty} B \log_2 \det \left(\mathbf{I}_{M_t} + \frac{\rho}{M_r} \mathbf{H}^H \mathbf{H} \right) = M_t B \log_2(1 + \rho).$$