EE359, Wireless Communications, Winter 2020
Homework 4 (100 pts)

Due: Friday Feb 7, 4 pm

Please refer to the homework page on the website (`ee359.stanford.edu/homework`) for guidelines.

1. (15 pts) **Waterfilling formula derivation** (Problem 4-12): Show using Lagrangian techniques that the optimal power allocation to maximize the capacity of a time-invariant block fading channel is given by the water-filling formula in Equation (4.27).

2. (15 pts) **Different notions of capacity** (Problems 4-4 and 4-5, adapted): Consider a flat fading channel of bandwidth 20 MHz and where, for a fixed transmit power $\bar{P}$, the received SNR is one of six values: $\gamma_1 = 20$ dB, $\gamma_2 = 15$ dB, $\gamma_3 = 10$ dB, $\gamma_4 = 5$ dB, $\gamma_5 = 0$ dB, and $\gamma_6 = -5$ dB. The probabilities associated with each state are $p_1 = p_6 = .1, p_2 = p_4 = .15,$ and $p_3 = p_5 = .25$.

   (a) (3 pts) Assume that only the receiver has CSI. Find the Shannon (ergodic) capacity of this channel.

   For the remaining subparts, assume that both the receiver and transmitter have CSI.

   (b) (5 pts) Give the optimal power adaptation policy for this channel and the corresponding Shannon capacity.

   (c) (3 pts) Determine the zero-outage capacity of this channel.

   (d) (4 pts) Determine the maximum outage capacity of this channel.

3. (10 pts) **Rayleigh fading and outage capacity** (Problem 4-8): In this problem we explore some properties of Rayleigh fading (with average received power 1, assuming a transmit power of 1). In particular, we investigate why we should not use channel inversion when the instantaneous gain γ is really low. In the following questions, we use the following fact for the Rayleigh pdf

   $$p(\gamma) = e^{-\gamma} > 1 - \gamma,$$

   where the right hand is the first order Taylor series expansion of $e^{-\gamma}$ around $\gamma = 0$.

   (a) (4 pts) For a small positive constant c and a smaller positive constant $\epsilon < c$, compute a lower bound for $\mathbf{E}\left[\frac{1}{\gamma}\right]$ by integrating the above lower bound within the interval $\gamma \in [\epsilon, c]$. This is a lower bound on the transmit power needed to maintain a constant received power. As $\epsilon \to 0$, show that the above expression is unbounded. This suggests that the transmit power needed to maintain a constant received power is unbounded.

   (b) (6 pts) Consider a probability density function $p(\gamma)$ whose value for $\gamma \in (0, c)$ is γ^l for a real number $l > -1$. Note that Rayleigh fading corresponds roughly to $l = 0$. For what values of l is $\mathbf{E}\left[\frac{1}{\gamma}\right]$ unbounded? Does this answer depend on the value of the probability density function over $\gamma \in (c, \infty)$?

4. (20 pts) **Maximize capacity through energy optimization** (Problem 4-7): A receiver stops at point A (100m from the cell phone tower) for 10 minutes and point B (500m from the cell phone tower) for 20 minutes. Assume the signal from the cell phone tower to the receiver only experiences path loss under the simplified path loss model with $\gamma = 3,\ d_0 = 1\text{m}$, and $K = 1$ along with AWGN, that the channel bandwidth is 20MHz, and the receiver noise PSD is $N_0 = 5 \times 10^{-14}$mW/Hz. Also, assume a fixed transmit power of 1mW.

   (a) (3 pts) What is the SNR received at points A and B?

   (b) (2 pts) What is total amount of energy (in Joules) that is transmitted during the two stops.

   (c) (5 pts) Assuming the transmitter sends data with a rate equal to the Shannon capacity of the channel at each of the stops, what is the total number of bits received?

   (d) (10 pts) Suppose now that the cell phone tower does not send at a fixed transmit power but fixes the total number of Joules it will allocate to the receiver at the two stops. Using the total amount of energy derived in part (b), determine the optimal allocation of this total energy at each of the stops to maximize the number of bits received. Assume that the transmitter sends data at a rate equal to the Shannon capacity with SNR based on your optimal energy allocation. How many bits are received with this optimal energy allocation and how does it compare to your answer in part (c)?

5. (20 pts) **More on shannon capacity** (Problem 4-9): Assume a Rayleigh fading channel, where the transmitter and receiver have CSI and the distribution of the fading SNR $p(\gamma)$ is exponential with mean $\bar{\gamma} = 10$ dB. Assume a channel bandwidth of 10 MHz. The `expint` command in MATLAB will be helpful for this question.

   (a) (5 pts) Find the cutoff value γ_0 and the corresponding power adaptation that achieves Shannon capacity on this channel.

   (b) (2 pts) Compute the Shannon capacity of this channel.

   (c) (2 pts) Compare your answer in part (b) with the channel capacity in AWGN with the same average SNR.

   (d) (3 pts) Compare your answer in part (b) with the Shannon capacity when only the receiver knows SNR γ.

   (e) (4 pts) Compare your answer in part (b) with the zero-outage capacity and outage capacity when the outage probability is .05.

   (f) (4 pts) Can a fading channel ever have a higher capacity than an AWGN channel with the same average SNR? Does this answer depend on whether or not the transmitter knows CSI? Give a short justification for your answers to both questions.

6. (20 pts) **Capacity with interference** (Problems 4-10 and 4-11): This problem illustrates the capacity gains that can be obtained from interference estimation and also how a malicious jammer can wreak havoc on link performance. Consider the interference channel depicted in Figure 1. The channel has a combination of AWGN $n[k]$ and interference $I[k]$. We model $I[k]$ as AWGN. The interferer is on (i.e., the switch is down) with probability .25 and off (i.e., switch up) with probability .75. The average transmit power is 10 mW, the noise PSD has $N_0 = 10^{-8}$ W/ Hz, the channel bandwidth B is 10 kHz (receiver noise power is N_0B), and the interference power (when on) is 9 mW.

   (a) (3 pts) What is the Shannon capacity of the channel if neither transmitter nor receiver know when the interferer is on (the interference power is known however) ?

   (b) (3 pts) Suppose the receiver knows when the interferer is on. What is the capacity of the chanel in this case?

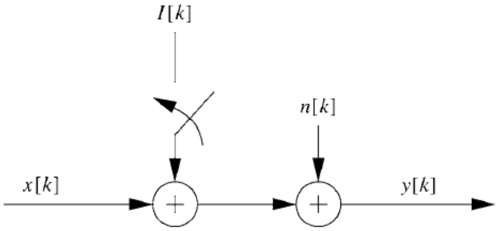

Figure 1: The channel in problem 6

(c) (4 pts) What is the capacity of the channel if both transmitter and receiver know when the interferer is on?

(d) (6 pts) Now, suppose that the jammer is always on, and that the transmitter knows the interference signal $I[k]$ perfectly. Consider two possible transmit strategies under this scenario: the transmitter can ignore the interference and use all its power for sending its signal, or it can use some of its power to cancel out the interferer (i.e. transmit the negative of the interference signal). In the first approach the interferer will degrade capacity by increasing the noise, and in the second strategy the interferer also degrades capacity since the transmitter sacrifices some power to cancel out the interference. Which strategy results in higher capacity?

Note: there is a third strategy, where the encoder actually exploits the structure of the interference in its encoding. This strategy is called dirty paper coding, and is used to achieve Shannon capacity on broadcast channels with multiple antennas.

(e) (4 pts) Suppose now that the interferer is a malicious jammer with perfect knowledge of $x[k]$ (so the interferer is no longer modeled as AWGN). Assume that neither transmitter nor receiver has knowledge of the jammer behavior. Assume also that the jammer is always on and has an average transmit power of 10 mW. What strategy should the jammer use to minimize the SNR of the received signal?