Sensing / Communication Application:

$X \longrightarrow y = hX + Z \longrightarrow N(\mu, \sigma^2)$ noise

y: observed signal

h: known coefficient

X: source signal

$$y_1 = h_1 X + Z_1 \sim N(\mu_1, \sigma_1^2)$$

$$y_2 = h_2 X + Z_2 .$$

$$y_n = h_n X + Z_n .$$

single source multiple sensor sensor

SIMO wireless channel.

$$y_1 = h_{11} X_1 + h_{12} X_2 + \dots h_{1M} X_M + Z_1$$

$$y_2 = h_{21} X_1 + h_{22} X_2 + \dots h_{2M} X_M + Z_2$$

$$y_n = h_{n1} X_1 + h_{n2} X_2 + \dots h_{nM} X_M + Z_M .$$

multi source multi-sensor system

MIMO wireless channels.

SIMO: $\bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $\bar{h} = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$ $\bar{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$

$$\bar{y} = \bar{h} X + \bar{Z}$$

MIMO: $\bar{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $\bar{h}_1 = \begin{bmatrix} h_{11} \\ h_{21} \\ \vdots \\ h_{n1} \end{bmatrix}$ $\bar{h}_2 = \begin{bmatrix} h_{12} \\ h_{22} \\ \vdots \\ h_{n2} \end{bmatrix}$ $\bar{h}_n = \begin{bmatrix} h_{1M} \\ h_{2M} \\ \vdots \\ h_{nM} \end{bmatrix}$

$$\bar{y} = \bar{h}_1 X_1 + \bar{h}_2 X_2 + \dots \bar{h}_M X_M + \bar{Z} \qquad \bar{Z} = \begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$$

$$H = \begin{bmatrix} | & | & & | \\ \bar{h}_1 & \bar{h}_2 & \cdots & \bar{h}_M \\ | & | & & | \end{bmatrix} \quad \bar{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix}$$

$$\underbrace{\bar{y}}_{\in \mathbb{R}^n} = \underbrace{H}_{\in \mathbb{R}^{n\times M}} \underbrace{\bar{X}}_{\in \mathbb{R}^M} + \underbrace{\bar{z}}_{\in \mathbb{R}^n}$$

Random vector

Def: An n-dimensional random vector $\bar{X} \in \mathbb{R}^n$ is a collection of n random variables expressed in the form

$$\bar{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

specified by $\sim$ the joint distribution of its entries

continious: $f_{X_1 X_2 \ldots X_n}(x_1, \ldots x_n)$

$\hookrightarrow$ joint pdf of $X_1, X_2 \ldots X_n$

discrete: $P_{X_1 X_2 \ldots X_M}(x_1, \ldots x_M)$

$\hookrightarrow$ joint pmf

$$f_{X_1 X_2}(x_1, x_2) = \iint_{x_3 \ldots x_n} f_{X_1 X_2 \ldots X_n}(x_1, x_2, x_3 \ldots x_n)\, dx_3 \ldots dx_n$$

marginal pdf of X_1 $\rightarrow$ $$f_{X_1}(x_1) = \iint_{x_2 \ldots x_n} f_{X_1 X_2 \ldots X_n}(x_1, \ldots x_n)\, dx_2 \ldots dx_n$$

$X \sim p_X(x) \quad f_X(x)$.

$$\mathbb{E}[X] = \mu \qquad Var(X) = \sigma^2$$

Def: The mean (vector) of a random vector $\bar{X}$ is the vector containing the mean of its entries.

$$\mathbb{E}[\bar{X}] = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix} \qquad \mathbb{E}\left(\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}\right) = \begin{bmatrix} \mathbb{E}[X_1] \\ \mathbb{E}[X_2] \\ \vdots \\ \mathbb{E}[X_n] \end{bmatrix}$$

How about "variance" of a random vector $\bar{X}$?

$$\bar{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad Var(X_i) \quad n \text{ terms}$$

$$Cov(X_i, X_j) = \mathbb{E}\left[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])\right]$$

$$Cov(X_i, X_i) = Var(X_i)$$

$\dfrac{n(n-1)}{2}$ covariance terms.

we have $n + \dfrac{n(n-1)}{2}$ second-order central moments.

Def: The covariance matrix of a random vector $\bar{X}$ is the following matrix

$$K_X = \begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & Cov(X_1,X_3) & \\ Cov(X_2,X_1) & Var(X_2) & Cov(X_2,X_3) & \\ Cov(X_3,X_1) & Cov(X_3,X_2) & Var(X_3) & \\ & & & \ddots \\ & & & & Var(X_n) \end{bmatrix}$$

$n + \dfrac{n^2 - n}{2}$

$n + \dfrac{n(n-1)}{2}$ distinct entries

$$(K_X)_{ij} = Cov(X_i, X_j).$$

$$(K_X)_{ij} = (K_X)_{ji} \implies K_X \text{ is symmetric}.$$

$$\underbrace{\underbrace{\left(\bar{X} - \mathbb{E}[\bar{X}]\right)}_{\in \mathbb{R}^{n\times 1}} \underbrace{\left(\bar{X} - \mathbb{E}[\bar{X}]\right)^T}_{\in \mathbb{R}^{1\times n}}}_{\text{outer product}}$$

$$\in \mathbb{R}^{n \times n}$$

$$\begin{bmatrix} X_1 - \mathbb{E}[X_1] \\ X_2 - \mathbb{E}[X_2] \\ \vdots \\ X_n - \mathbb{E}[X_n] \end{bmatrix} \begin{bmatrix} X_1 - \mathbb{E}[X_1] & X_2 - \mathbb{E}[X_2] & \cdots & X_n - \mathbb{E}[X_n] \end{bmatrix}$$

$$= \begin{bmatrix} (X_1 - \mathbb{E}[X_1])(X_1 - \mathbb{E}[X_1]) & (X_1 - \mathbb{E}[X_1])(X_2 - \mathbb{E}[X_2]) & \cdots \\ (X_2 - \mathbb{E}[X_2])(X_1 - \mathbb{E}[X_1]) & (X_2 - \mathbb{E}[X_2])(X_2 - \mathbb{E}[X_2]) & \cdots \\ \vdots & & \\ & & (X_n - \mathbb{E}[X_n])(X_n - \mathbb{E}[X_n]) \end{bmatrix}$$

random matrix: a matrix whose entries are random variables.

Extension: The $\mathbb{E}[M]$ where $M \in \mathbb{R}^{m \times n}$ with random entries M_{ij} is defined as

$$\mathbb{E}[M] = \begin{bmatrix} \mathbb{E}[M_{11}] & \mathbb{E}[M_{12}] & \cdots \\ \mathbb{E}[M_{21}] & \mathbb{E}[M_{22}] & \\ \vdots & & \end{bmatrix}$$

$$\left(\mathbb{E}[M]\right)_{ij} = \mathbb{E}[M_{ij}]$$

The covariance matrix K_X of a random vector X is

$$K_{\bar{X}} = \mathbb{E}\left[\left(\bar{X} - \mathbb{E}[\bar{X}]\right)\left(\bar{X} - \mathbb{E}[\bar{X}]\right)^T \right].$$

Note: $K_{\bar{X}} = \mathbb{E}\left[\underbrace{\left(\bar{X} - \mathbb{E}[\bar{X}]\right)}_{\in \mathbb{R}^{n\times 1}} \underbrace{\left(\bar{X} - \mathbb{E}[\bar{X}]\right)}_{\in \mathbb{R}^{n\times 1}} \right]$ wrong. does not exist.

Ex: Linear transformations of a random vector $\bar{X}$

$$\underbrace{\bar{Y}}_{\in \mathbb{R}^{m\times 1}} = \underbrace{H}_{\text{fixed matrix, deterministic } \in \mathbb{R}^{m\times n}} \underbrace{\bar{X}}_{\in \mathbb{R}^{n\times 1}} . \qquad \mathbb{E}[\bar{X}] = \bar{\mu},\ K_{\bar{X}}$$

$$\mathbb{E}[\bar{Y}] = \mathbb{E}\left(\begin{bmatrix} \left(\sum_{i=1}^{n} H_{1i} X_i\right) \\ \left(\sum_{i=1}^{n} H_{2i} X_i\right) \\ \vdots \\ \left(\sum_{i=1}^{n} H_{ni} X_i\right) \end{bmatrix} \right) = \begin{bmatrix} \sum_{i=1}^{n} H_{1i}\, \mathbb{E}[X_i] \\ \sum_{i=1}^{n} H_{2i}\, \mathbb{E}[X_i] \\ \vdots \\ \sum_{i=1}^{n} H_{ni}\, \mathbb{E}[X_i] \end{bmatrix}.$$

$$= H\,\mathbb{E}[\bar{X}].$$

Fact: If $\bar{Y} = H\bar{X} + \bar{Z}$ $\mathbb{E}[\bar{Y}] = H\,\mathbb{E}[\bar{X}] + \mathbb{E}[\bar{Z}]$.

Linearity of expectation.

analogous to $\mathbb{E}[aX + b] = a\,\mathbb{E}[X] + b$.

for a random variable X.

M is a random matrix.

$$\underbrace{A}_{\in \mathbb{R}^{n\times n}} = \underbrace{H}_{\substack{\in \mathbb{R}^{m\times n} \\ \text{deterministic}}} \underbrace{M}_{\in \mathbb{R}^{n\times n}\ \text{random matrix}} \Rightarrow \mathbb{E}[A] = H\,\mathbb{E}[M].$$

random matrix.

Prove by direct computation.

$$\bar{Y} = H\bar{X}$$

$$K_{\bar{Y}} = \mathbb{E}\left[\left(\bar{Y} - \mathbb{E}[\bar{Y}]\right)\left(\bar{Y} - \mathbb{E}[\bar{Y}]\right)^T\right]$$

$$= \mathbb{E}\left[\left(H\bar{X} - \mathbb{E}[H\bar{X}]\right)\left(H\bar{X} - \mathbb{E}[H\bar{X}]\right)^T\right]$$

$$= \mathbb{E}\left[\left(H\bar{X} - H\mathbb{E}[\bar{X}]\right)\left(H\bar{X} - H\mathbb{E}[\bar{X}]\right)^T\right]$$

$$= \mathbb{E}\left[H\left(\bar{X} - \mathbb{E}[\bar{X}]\right)\left(H\left(\bar{X} - \mathbb{E}[\bar{X}]\right)\right)^T\right]$$

$(AB)^T = B^T A^T$

$$= \mathbb{E}\left[H\left(\bar{X} - \mathbb{E}[\bar{X}]\right)\left(\bar{X} - \mathbb{E}[\bar{X}]\right)^T H^T\right]$$

$$= H\,\mathbb{E}\left[\left(X - \mathbb{E}[\bar{X}]\right)\left(X - \mathbb{E}[\bar{X}]\right)^T H^T\right]$$

$$= H\,\underbrace{\mathbb{E}\left[\left(X - \mathbb{E}[\bar{X}]\right)\left(\bar{X} - \mathbb{E}[\bar{X}]\right)^T\right]}_{K_X} H^T$$

$$K_{\bar{Y}} = H K_X H^T$$

analogous $\quad y = bX$

$$Var(y) = b^2\, Var(X) = b\, Var(X)\, b$$