Kalman Filtering

Linear Dynamical System:

$$X_0 \sim \mathcal{N}(0, \sigma_0^2) \quad \text{initial state}$$

$$X_n = \alpha X_{n-1} + V_n \qquad V_n \sim \mathcal{N}(0, \sigma_V^2)$$

$$y_n = X_n + Z_n \qquad Z_n \sim \mathcal{N}(0, \sigma_Z^2)$$

$X_0, V_1, \ldots V_n, Z_1, \ldots Z_n$ are all independent of each other

n = 0

$$X_0 \sim \mathcal{N}(0, \sigma_0^2)$$

$$y_0 = X_0 + Z_0$$

$$\hat{X}_0 = \frac{\sigma_0^2}{\sigma_0^2 + \sigma_Z^2} y_0 \qquad e_0^2 = \frac{\sigma_0^2 \sigma_Z^2}{\sigma_0^2 + \sigma_Z^2}$$

n = 1 :

$$X_1 = \alpha X_0 + V_1$$

$$y_1 = X_1 + Z_1$$

$\hat{X}_0, \cancel{y_0}$ $\qquad X_0 = \hat{X}_0 + W_0 \qquad W_0 \sim \mathcal{N}(0, e_0^2)$

y_0 independent of W_0

$$X_1 = \alpha \hat{X}_0 + \alpha W_0 + V_1 \sim \mathcal{N}(\alpha \hat{X}_0, \alpha^2 e_0^2 + \sigma_V^2)$$

$$y_1 = X_1 + Z_1 \qquad Z_1 \sim \mathcal{N}(0, \sigma_Z^2)$$

$$\hat{X}_1 = \alpha \hat{X}_0 + \overbrace{\frac{\alpha^2 e_0^2 + \sigma_V^2}{\alpha^2 e_0^2 + \sigma_V^2 + \sigma_Z^2}}^{= k_1} (y_1 - \alpha \hat{X}_0)$$

$$= \alpha (1 - k_1) \hat{X}_0 + k_1 y_1$$

$$e_1 = k_1 \sigma_1^2 .$$

$n-1$:

$$X_n = \alpha X_{n-1} + V_n .$$

$$y_n = X_n + Z_n$$

previous step : $\hat{X}_{n-1}$, y_{n-1} and we know e_{n-1}^2, k_{n-1}

$$X_{n-1} = \hat{X}_{n-1} + W_{n-1} \quad W_{n-1} \sim N(0, e_{n-1}^2) .$$

y_{n-1} independent of W_{n-1}

$$X_n = \alpha \hat{X}_{n-1} + \alpha W_{n-1} + V_n \sim N(\alpha \hat{X}_{n-1}, \alpha^2 e_{n-1}^2 + \sigma_V^2)$$

$$y_n = X_n + Z_n . \quad Z_n \sim N(0, \sigma_Z^2)$$

$$\hat{X}_n = \alpha \hat{X}_{n-1} + \underbrace{\frac{\alpha^2 e_{n-1}^2 + \sigma_V^2}{\alpha^2 e_{n-1}^2 + \sigma_V^2 + \sigma_Z^2}}_{= k_n} (y_n - \alpha \hat{X}_{n-1})$$

$$= \alpha (1 - k_n) \hat{X}_{n-1} + k_n y_n .$$

$$e_n^2 = k_n \sigma_Z^2 .$$

Kalman filter Recursion.

Take : $\hat{X}_{n-1}$ and e_{n-1}^2 from previous step

Compute : $k_n = \dfrac{\alpha^2 e_{n-1}^2 + \sigma_V^2}{\alpha^2 e_{n-1}^2 + \sigma_V^2 + \sigma_Z^2}$

$$\hat{X}_n = \alpha (1 - k_n) \hat{X}_{n-1} + k_n y_n$$

$$e_n^2 = k_n \sigma_Z^2 .$$

andom processes:

Def: A random process $\{X(t) : t \in \mathcal{J}\}$ is an infinite collection of random variables.

If t is discrete, $\mathcal{J} = \mathbb{Z}$ is countable, then $X(t)$ is called a discrete-time process. We will switch our notation to $X[n]$ or X_n for $n \in \mathbb{Z}$.

If t is continious, $\mathcal{J} = \mathbb{R}$, $X(t)$ is a continuous-time process.

Examples: Electromagnetic signal captured by a receive antenna.

Voltage at a certain point in our circuit.

Daily price of a stock.

Sequence of bits.

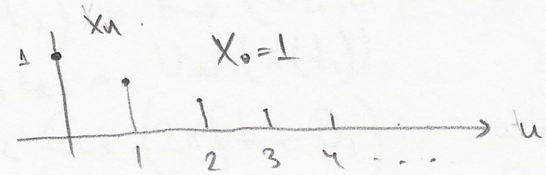
Each realization of a random process is a sample function or a sample sequence.

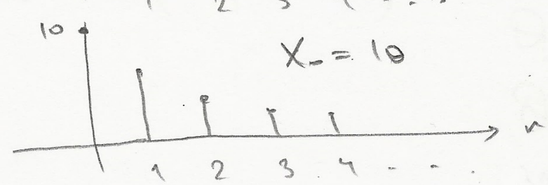
Example: State of a dynamical system over time

$X_0, X_1, X_2, \ldots$

$$X_n = \alpha X_{n-1} + V_n \qquad X_0 \sim \mathcal{N}(0, \sigma_0^2)$$

$\cdot\ V_n = 0$, $\alpha < 1$.

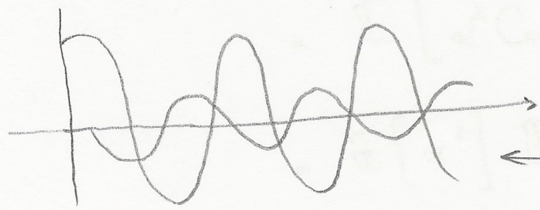




Example:

$$X(t) = \alpha \cos(2\pi f t + \theta)$$

f is fixed
$\alpha \sim \mathcal{N}(0, \sigma^2)$
$\theta \sim$ unif $[0, 2\pi]$.



possible realizations (sample functions) for this process.

A random process $X(t)$ is specified by specifying the joint distribution (pmf, pdf, cdf) of the random variables

$$X(t_1) \quad X(t_2) \quad \dots \quad X(t_n).$$

for any n and any time instants $t_1, t_2, \dots t_n \in \mathcal{T}$.

Def: The mean of a random process $X(t)$ is the function $\mu_X(t) = \mathbb{E}\{X(t)\}$

Def: The autocovariance function of a random process

$$K_X(t_1, t_2) = \mathbb{E}\Big[\big(X(t_1) - \underbrace{\mathbb{E}[X(t_1)]}_{\mu_X(t_1)}\big)\big(X(t_2) - \underbrace{\mathbb{E}[X(t_2)]}_{\mu_X(t_2)}\big)\Big]$$

Note: $K_X(t_1, t_2) = K_X(t_2, t_1)$
$K_X(t_1, t_1) = \mathrm{Var}(X(t_1))$

Example: $X(t) = \alpha \cos(2\pi f t + \theta)$

$\alpha \sim \mathcal{N}(0, \sigma^2)$
$\theta \sim$ unif $[0, 2\pi]$
θ and α are independent

$$\mu_X(t) = \mathbb{E}[X(t)] = \mathbb{E}[\alpha \cos(2\pi f t + \theta)]$$
$$= \mathbb{E}[\alpha]\,\mathbb{E}[\cos(2\pi f t + \theta)] = 0$$

$$K_X(t_1, t_2) = \mathbb{E}[X(t_1) X(t_2)]$$
$$= \mathbb{E}[\alpha^2 \cos(2\pi f t_1 + \theta) \cos(2\pi f t_2 + \theta)]$$
$$= \mathbb{E}[\alpha^2]\,\mathbb{E}[\cos(2\pi f t_1 + \theta)\cos(2\pi f t_2 + \theta)]$$
$$= \sigma^2 \int_0^{2\pi} \cos(2\pi f t_1 + \theta) \cos(2\pi f t_2 + \theta) \frac{1}{2\pi}\, d\theta$$

Use $\cos x \cos y = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(x-y)$.

$$= \sigma^2 \cos(2\pi f (t_1 - t_2))$$

<u>Gaussian Process:</u> $\{X(t) : t \in \mathcal{T}\}$ is a Gaussian process iff for all n and $t_1, t_2, \ldots \in \mathcal{T}$

$X(t_1)\ X(t_2) \ldots X(t_n)$ are jointly Gaussian

<u>Example:</u> $X_n = \alpha X_{n-1} + V_n \qquad X_0 \sim \mathcal{N}(0, \sigma_0^2)$

$V_n \sim \mathcal{N}(0, \sigma_Z^2)$ independ

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ \alpha & 1 & 0 & \cdots \\ \alpha^2 & \alpha & 1 & 0 \cdots \\ \alpha^n & \alpha^{n-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ V_1 \\ \vdots \\ V_n \end{bmatrix}$$

$\to$ Gaussian random vector because $X_0, V_1, \ldots V_n$ are Gaussian r.v's and independ

$\{X_n, n \in \mathbb{Z}\}$ is a Gaussian process.