EE 278: Probability and Statistical Inference:

1) A second course in probability

2) Introduction to Statistical Inference

Probability

A) Concentration Inequalities / High-dimensional probabili

EE178: Law of Large Numbers

Ex: $X_1, X_2, \dots X_n$ iid $\text{Unif}[0,1]$.

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} \mathbb{E}[X_i] = \frac{1}{2}.$$

Ex: Every student in this class attends exactly! % 50 of the lectures in person.

Ex: four lectures $\{1,2,3,4\}$: $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,$

What is the probability that a given student attends the k'th lecture? Assume there are m lectures

$$p = \frac{\binom{m-1}{m-1-(\frac{m}{2}-1)}}{\binom{m}{m/2}} = \frac{\frac{(m-1)!}{\cancel{(m/2)!}\,(m/2-1)!}}{\frac{m!}{\cancel{(m/2)!}\,(m/2)!}} = \frac{1}{m}\, m/2 = \frac{1}{2}$$

Alternatively, define

$$\underset{\substack{\uparrow \quad \searrow \\ \text{student} \quad \text{lecture}}}{X_{i,k}} = \begin{cases} 1 & \text{if the i'th student attends the k'th lecture} \\ 0 & \text{otherwise} \end{cases}$$

$\mathbb{E}[X_{i,k}] = p$: the probability that the i'th student attends the k'th lecture

$$X_{i,1} + X_{i,2} + X_{i,3} + \dots + X_{i,m} = \frac{m}{2}$$

$$\mathbb{E}\left[X_{i,1} + X_{i,2} + X_{i,3} + \dots + X_{i,m}\right] = \frac{m}{2}$$

$$\underbrace{\mathbb{E}\left[X_{i,1}\right]}_{p} + \underbrace{\mathbb{E}\left[X_{i,2}\right]}_{p} + \mathbb{E}\left[X_{i,3}\right] + \dots + \mathbb{E}\left[X_{i,m}\right] = \frac{m}{2}$$

$$\Rightarrow mp = \frac{m}{2} \Rightarrow p = \frac{1}{2}$$

$$\sum_{i=1}^{60} X_{i,9}$$: # of number of students that show up for the 9'th lecture

LLN $$\frac{1}{60}\sum_{i=1}^{60} X_{i,9} \longrightarrow \mathbb{E}\left[X_{i,9}\right] = p = \frac{1}{2}$$

$$\sum_{i=1}^{60} X_{i,9} \approx 30$$

For a class size of $n = 60$, how resonable it is to expect the # of students that attend a particular lecture to be close to 30, e.g. $[25, 35]$.

B) Random vectors : $\overline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ → a random variable

- Gaussian random vectors
- Covariance matrix (principal component analysis & dimensionality reduction)

c) Random process : $X[t]$ $t = 1, \dots \infty$.

$X(t)$: $t \in \mathbb{R}$

- Basic concepts : Stationarity, Covariance function
- Gaussian processes, White noise

2) Inferece Problems

$$X \longrightarrow \boxed{\text{noisy channel}} \longrightarrow y \longrightarrow \boxed{\text{predictor}} \longrightarrow \hat{X}$$

X: State of a physical system
Communicated message
Label of an image, e.g. cat or dog
State of the patiat

A) X is discrete $X \in \{0,1\}$ a bit
$X \in \{\text{cat}, \text{dog}\}$
$X \in \{\text{healthy}, \text{infected}\}$
$X \in \{\text{fire}, \text{no fire}\}$

Hypothesis Testing | Detection | Decoding | Classificatio
Decision - making

Performace Criteria: Probability of error $P(\hat{X} \neq X)$

Find the best predictor that minimizes the probability of error.

B) X is a continious random variable

X: location of a robot arm
X: the value of a stock

Estimation problems

$$P(\hat{X} \neq X) = 1$$

$\ell(x, \hat{x})$: cost | loss | error fuction
$\ell(x, \hat{x}): \mathbb{R}^2 \longrightarrow \mathbb{R}^+$

Find a predictor s.t. $\mathbb{E}\left[\ell(X, \hat{X})\right]$ is minimized.

Probability of error is a special case

$$\ell(x,\hat{x}) = \begin{cases} 1 & \text{if } \hat{x} \neq x \\ 0 & \text{o/w} \end{cases}$$

Popular loss function: $\ell(\hat{x},x) = \underbrace{(x-\hat{x})^2}_{\text{squared loss}}$

Minimum mean-squared estimation

I) X - random variable Y - random variable

II) X - random vector Y - random vector

III) X - random process Y - random process

Model-based Approach | Data-based Approach (Machine Learning Approach)

1) data $\rightarrow$ model $\rightarrow$ predictor
physics $\nearrow$

data $\rightarrow$ predictor

Eg. $Y = \alpha X + W$ ← noise

Eg. $\{(X_i, Y_i)_{i=1}^{L}\} \rightarrow \{f_i, i=1..M\}$

labels images

2) linear models $\rightarrow$ optimal linear predictors

non-linear predictors e.g. neural networks

3) quantify performance analytically

cross-validation